Nonmonotonic Upper Probabilities and Quantum Entanglement

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Abstract

A well-known property of quantum entanglement phenomena is that random variables representing the observables in a given experiment do not have a joint probability distribution. The main point of this lecture is to show how a generalized distribution, which is a nonmonotonic upper probability distribution, can be used for all the observables in two important entanglement cases: the four random variables or observables used in Bell-type experiments and the six correlated spin observables in three-particle GHZ-type experiments. Whether or not such upper probabilities can play a significant role in the conceptual foundations of quantum entanglement will be discussed.

Definition 1 Let Ω be a nonempty set, $\mathcal{F}$ a Boolean algebra on Ω, and P^* a real-valued function on $\mathcal{F}$. Then $\mathbf{\Omega} = (\Omega, \mathcal{F}, P^*)$ is an *upper probability space* if and only if for every A and B in $\mathcal{F}$

1. $0 \leq P^*(A) \leq 1$;
2. $P^*(\emptyset) = 0$ and $P^*(\Omega) = 1$;
3. If $A \cap B = \emptyset$, then $P^*(A \cup B) \leq P^*(A) + P^*(B)$.

Moreover, P^* is *monotonic* if and only if whenever $A \subseteq B$

$$P^*(A) \leq P^*(B).$$

Theorem 1 Joint Distribution Theorem. Let $\mathbf{X}, \mathbf{Y}$, and $\mathbf{Z}$ be random variables with possible values 1 and -1, and with

$$E(\mathbf{X}) = E(\mathbf{Y}) = E(\mathbf{Z}) = 0$$

Then a necessary and sufficient condition for the existence of a joint probability distribution of the three random variables is that the following two inequalities be satisfied.

$$-1 \leq E(\mathbf{XY}) + E(\mathbf{YZ}) + E(\mathbf{XZ}) \leq 1$$
$$+2\min\{E(\mathbf{XY}), E(\mathbf{YZ}), E(\mathbf{XZ})\}.$$

Corollary 1 In the symmetric case, where

$$E(\mathbf{XY}) = E(\mathbf{YZ}) = E(\mathbf{XZ}),$$

the inequalities simplify to

$$-\frac{1}{3} \leq E(\mathbf{XY}) \leq 1.$$

Consider three random variables $\mathbf{X}_1$, $\mathbf{X}_2, \mathbf{X}_3$ with values ± 1 and expectations

$$E(\mathbf{X}_1) = E(\mathbf{X}_2) = E(\mathbf{X}_3) = 0$$
$$\mathrm{Cov}(\mathbf{X}_i, \mathbf{X}_j) = -1, \quad i \neq j.$$

We use the notation

$$p_{i\bar{j}} = P(\mathbf{X}_i = 1, \mathbf{X}_j = -1), \text{ etc.}$$

So

$$p_{i\bar{j}} = p_{\bar{i}j} = \frac{1}{2}, \quad i \neq j$$
$$p_{ij} = p_{\bar{i}\bar{j}} = 0.$$

This implies, to fit the correlations,

$$p^*_{i\bar{j}} = \tfrac{1}{2},\ p^*_{\bar{i}j} = \tfrac{1}{2}$$
$$p^*_{ij} = 0,\ p^*_{\bar{i}\bar{j}} = 0.$$

Note that

$$p^*_{i\bar{j}} = P^*(\mathbf{X}_i = 1, \mathbf{X}_j = -1).$$

Since "mixed" $i\bar{j}$ or $\bar{i}j$ never occur in p^*_{123} or $p^*_{\overline{123}}$, we may set

$$p^*_{123} = p^*_{\overline{123}} = 0.$$

By symmetry and to satisfy subadditivity—e.g., $p^*_{1\bar{2}} \leq p^*_{1\bar{2}3} + p^*_{\overline{123}}$, since

$$p^*_{i\bar{j}} = p^*_{\bar{i}j} = \frac{1}{2}, \qquad \text{for} \quad i \neq j$$

we set the remaining 6 triples at $\frac{1}{4}$:

$$p^*_{12\bar{3}} = p^*_{1\bar{2}3} = p^*_{\bar{1}23} = p^*_{1\overline{23}} = p^*_{\bar{1}2\bar{3}} = p^*_{\overline{12}3} = \frac{1}{4}.$$

Notice that P^* is nonmonotonic for $p^*_{12\bar{3}} > p_{12} = 0$.

Theorem 2 Theorem on Common Causes. Let $\mathbf{X}_1 \ldots \mathbf{X}_n$ be two-valued random variables. Then a necessary and sufficient condition that there is a random variable $\boldsymbol{\lambda}$ such that $\mathbf{X}_1 \ldots \mathbf{X}_n$ are conditionally independent given $\boldsymbol{\lambda}$ is that there exists a joint probability distribution of $\mathbf{X}_1 \ldots \mathbf{X}_n$.

The random variable $\boldsymbol{\lambda}$ would be called a *hidden variable* in quantum mechanics.

Let $\mathbf{\Omega} = (\Omega, \mathcal{F}, P^*)$ be an upper probability space and let λ be a function from Ω to Re^k such that for every vector $(b_1, \ldots, b_k)$ the set

$$\{\omega : \omega \in \Omega \ \& \ \lambda_i(\omega) \leq b_i, = 1, \ldots, k\}$$

is in $\mathcal{F}$. Then λ is a *generalized random variable* (*with respect to* $\mathbf{\Omega}$).

Theorem 3 Generalized Common Causes. Let $\mathbf{X}_1, \ldots, \mathbf{X}_n$ be two-valued (± 1) random variables whose common domain is a space Ω with an algebra $\mathcal{F}$ of events that includes the subalgebra $\mathcal{F}^*$ of cylinder sets of dimension n defined above. Also, let pairwise probability functions $P_{ij}, 1 \leq i < j \leq n$, compatible with the single functions $P_i, 1 \leq i \leq n$, be given. Then there exists an upper probability space $\mathbf{\Omega} = (\Omega, \mathcal{F}^*, P^*)$, and a generalized random variable λ on Ω to the set of n-dimensional vectors whose components are ± 1 such that for $1 \leq i < j \leq n$ and every value λ of $\boldsymbol{\lambda}$:

(i) $P^*(\mathbf{X}_i = \pm 1, \mathbf{X}_j = \pm 1) = P_{ij}(\mathbf{X}_i = \pm 1, \mathbf{X}_j = \pm 1)$;

(ii) $P^*(\mathbf{X}_1 = \lambda_1, \ldots, \mathbf{X}_n = \lambda_n) = P^*(\boldsymbol{\lambda}_1 = \lambda_1, \ldots, \boldsymbol{\lambda}_n = \lambda_n)$;

(iii) $\boldsymbol{\lambda}$ is *deterministic*, i.e.,

$$P(\mathbf{X}_i = 1 | \boldsymbol{\lambda}_i = 1) = 1$$

and

$$P(\mathbf{X}_i = -1 | \boldsymbol{\lambda}_i = -1) = 1$$

(iv)

$$E(X_i X_j | \boldsymbol{\lambda} = \lambda)$$
$$= E(X_i | \boldsymbol{\lambda} = \lambda) E(X_i | \boldsymbol{\lambda} = \lambda).$$

Theorem 4 Monotonicity Implies Probability. Let $\mathbf{X}_1, \mathbf{X}_2$ and $\mathbf{X}_3$ be two-valued ± 1 random variables with $E(\mathbf{X}_i) = 0, i = 1, 2, 3$, such that there is a monotonic upper probability function compatible with the given correlations $E(\mathbf{X}_i, \mathbf{X}_j), 1 \leq i < j \leq 3$. Then there exists a joint probability distribution of $\mathbf{X}_1, \mathbf{X}_2$ and $\mathbf{X}_3$ compatible with the given means and correlations.

Theorem 5 Nonmonotonicity. Let $\mathbf{X}_1$, $\mathbf{X}_2$ and $\mathbf{X}_3$ be two-valued (± 1) random variables with $E(X_i) = 0$, $i = 1, 2, 3$, such that there is no joint probability distribution compatible with the correlations $E(\mathbf{X}_i, \mathbf{X}_j), 1 \leq i < j \leq 3$. Then any upper measure P^* compatible with the given means and correlations cannot satisfy the axiom of monotonicity.

Theorem 6 Let $(\Omega, \mathcal{F}, P^*)$ be an upper probability space such that P^* is nonmonotonic. Then the lower probability defined by

$$P_*(A) = 1 - P^*(A)$$

is not superadditive. So P^* is not a proper lower probability.

Quantum Mechanics

Measuring Apparatus

A **B**

singlet source

angle $\angle \mathbf{AB} = \theta$

The results may be most easily discussed in terms of a system of two spin-$\frac{1}{2}$ particles initially in the singlet state.

Qualitative Axioms Assumed about Measurements and Hidden Variables

1. *Axial symmetry.* For any direction of the measuring apparatus the expected spin is 0, where spin is measured by +1 and -1 for spin $+\frac{1}{2}$ and spin $-\frac{1}{2}$, respectively. Further, the expected product of the spin measurements is the same for different orientations of the measuring apparatuses, as long as the angle between the measuring apparatuses remains the same.

2. *Opposite measurement for same orientation.* The correlation between the spin measurements is -1 if the two measuring apparatuses ahve the same orientation.

3. *Independence of* $\boldsymbol{\lambda}$. The expectation of any function of λ is independent of the orientation of the measuring apparatus.

4. *Locality.* The spin measurement obtained with one apparatus is independent of the orientation of the other measuring apparatus.

5. *Determinism.* Given $\boldsymbol{\lambda}$ and the orientation of the measuring apparatus, the results of the two spin measurements are conditionally statistically independent.

For example of Axiom 5. Conditional statistical independence

$$E(AB|\lambda) = E(A|\lambda)E(B|\lambda).$$

Quantum Mechanics

$$\text{Covariance}(\mathbf{AB}) = \mathbf{AB} = -\cos\theta$$

where θ is angle difference of orientation of $\mathbf{A}$ and $\mathbf{B}$.

Bell Inequalities

$$\begin{aligned}
-2 &\leq \mathbf{AB} + \mathbf{AB'} + \mathbf{A'B} - \mathbf{A'B'} \leq 2 \\
-2 &\leq \mathbf{AB} + \mathbf{AB'} - \mathbf{A'B} + \mathbf{A'B'} \leq 2 \\
-2 &\leq \mathbf{AB} - \mathbf{AB'} + \mathbf{A'B} + \mathbf{A'B'} \leq 2 \\
-2 &\leq -\mathbf{AB} + \mathbf{AB'} + \mathbf{A'B} + \mathbf{A'B'} \leq 2
\end{aligned}$$

Theorem 7 Bell's inequalities in the above Clauser, Horn, Shimony and Holt (1969) form are necessary and sufficient for the random variables $\mathbf{A}, \mathbf{A}', \mathbf{B}$ and $\mathbf{B}'$ to have a joint probability distribution compatible with the given covariances.

Quantum mechanics does not satisfy these inequalities in general. To illustrate ideas, we take as a particular case the following:

$$\mathbf{AB} - \mathbf{AB'} + \mathbf{A'B} + \mathbf{A'B'} < -2.$$

We choose

$$\begin{aligned}
\mathbf{AB} = \mathbf{A'B'} &= -\cos 30^\circ = -\frac{\sqrt{3}}{2} \\
\mathbf{AB'} &= -\cos 60^\circ = -\frac{1}{2} \\
\mathbf{A'B} &= -\cos 0^\circ = -1.
\end{aligned}$$

So

$$-\frac{\sqrt{3}}{2} + \frac{1}{2} - 1 - \frac{\sqrt{3}}{2} < -2.$$

Theorem 8 Existence of Hidden Variables. Let $\mathbf{AB}, \mathbf{AB'}$, $\mathbf{A'B}$ and $\mathbf{A'B'}$ be any four quantum mechanical covariances, which will in general not satisfy the Bell inequalities. Then there is an upper probability P^* consistent with the given covariances and a generalized hidden variable $\boldsymbol{\lambda}$ with P^* such that, for every value λ of $\boldsymbol{\lambda}$,

$$E(\mathbf{AB}|\boldsymbol{\lambda}=\lambda)=E(\mathbf{A}|\boldsymbol{\lambda}=\lambda)E(\mathbf{B}|\boldsymbol{\lambda}=\lambda)$$

and similarly for $\mathbf{AB'}, \mathbf{A'B}$ and $\mathbf{A'B'}$.

Theorem 9 Monotonicity Implies Bell Inequalities. Let $\mathbf{A}, \mathbf{A}', \mathbf{B}$, and $\mathbf{B}'$ be two-valued (± 1) random variables with expectation $E(\mathbf{A}) = E(\mathbf{A}') = E(\mathbf{B}) = E(\mathbf{B}') = 0$ such that there is a monotonic upper probability function compatible with the given correlations $\mathbf{AB}, \mathbf{AB}', \mathbf{A}'\mathbf{B}$, and $\mathbf{A}'\mathbf{B}'$. Then the given covariances satisfy the Bell inequalities.

Three-particle Entanglement

But first some pure probability.

Theorem 10 Let $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ be random variables with values ± 1. Then there is no probability distribution to support the following expectations:

(i) $E(\mathbf{A}) = E(\mathbf{B}) = E(\mathbf{C}) = 1$,

(ii) $E(ABC) = -1$.

But there is a nonmonotonic upper probability P^* that does.

Sketch of Proof:

$$E(\mathbf{A}) = p(a..) - p(\overline{a}..)$$

Similarly for $E(B)$ and $E(C)$.
Notation $p(a) = p(a..)$, etc.
So we set:

$$p(a) = p(b) = p(c) = 1$$
$$p(\overline{a}) = p(\overline{b}) = p(\overline{c}) = 0$$

$$
\begin{array}{rll}
p(a) & \leq & p^*(ab) + p^*(a\overline{b}) \\
& \leq & (p^*(abc) + p^*(ab\overline{c})) + (p^*(a\overline{b}c) + p^*(a\overline{bc})) \\
1 & & (1 \quad + \quad \frac{1}{3}) \quad + \quad (\frac{1}{3} \quad + \quad 0)
\end{array}
$$

Simplifying notation further:

$$abc = p^*(abc), \text{ etc.}$$

$$
\begin{array}{rll}
E(ABC) & = & (abc + a\overline{bc} + \overline{a}b\overline{c} + \overline{ab}c) \\
& & (\ 1 \quad + \quad 0 \quad + \quad 0 \quad + \quad 0) \\
& & -(\overline{abc} + ab\overline{c} + a\overline{b}c + \overline{a}bc) \\
& & -(1 \ + \ \frac{1}{3} \ +; \ \frac{1}{3} \ + \ \frac{1}{3}) \\
& = & -1
\end{array}
$$

Note strong nonmonotonicity:

$$p^*(\overline{a}) = 0 < 1 = p^*(\overline{abc})$$

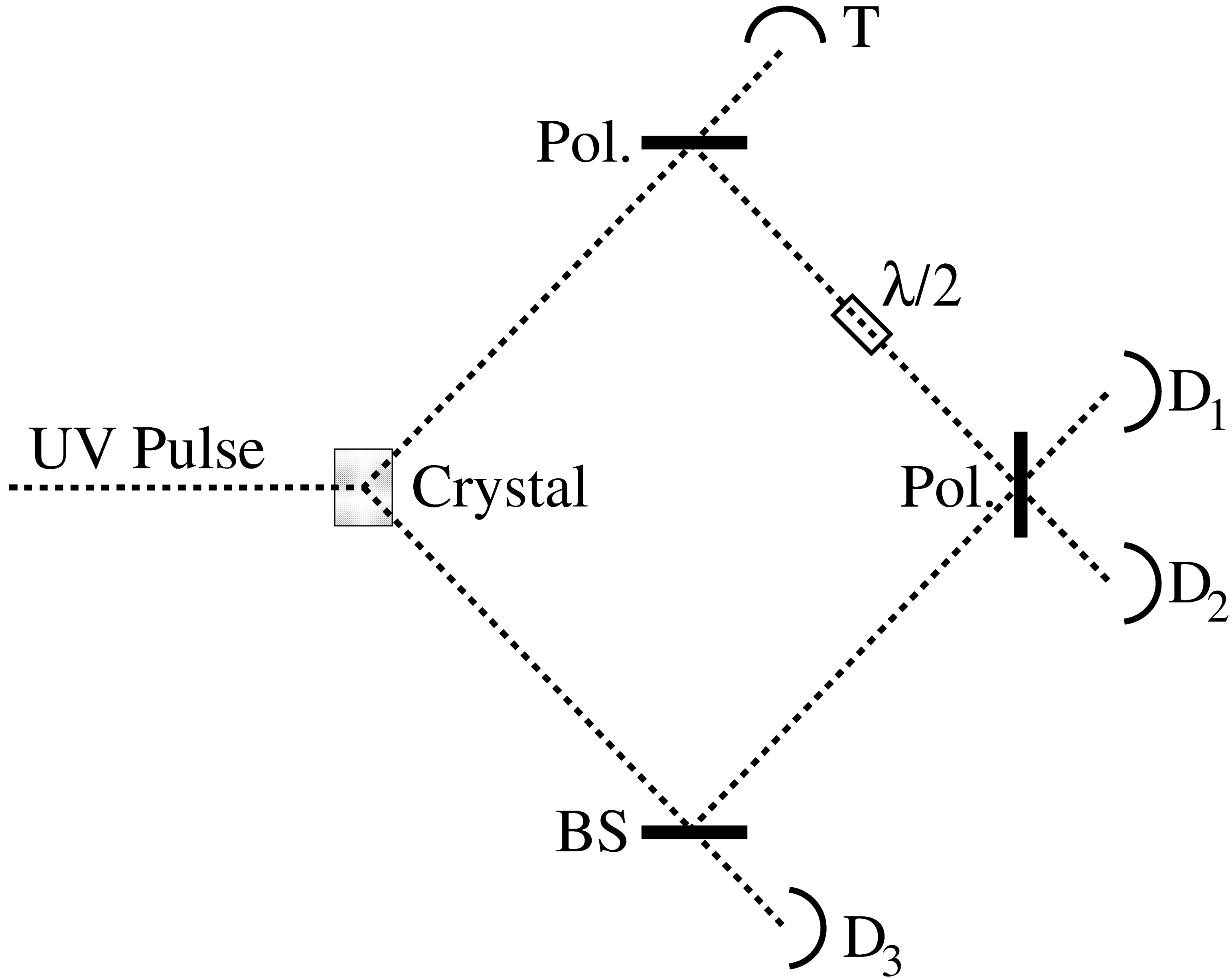

FIG. 1

Fig. 1. Scheme for the Innsbruck GHZ experiment. The GHZ correlations are obtained when all detectors T, D_1, D_2, and D_3 register a photon within the same window of time.

GHZ

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|+++\rangle + |---\rangle), \qquad (1)$$

$$\hat{A}|\psi\rangle = \hat{\sigma}_{1x}\hat{\sigma}_{2y}\hat{\sigma}_{3y}|\psi\rangle = |\psi,\rangle \qquad (2)$$

$$\hat{B}|\psi\rangle = \hat{\sigma}_{1y}\hat{\sigma}_{2x}\hat{\sigma}_{3y}|\psi\rangle = |\psi,\rangle \qquad (3)$$

$$\hat{C}|\psi\rangle = \hat{\sigma}_{1y}\hat{\sigma}_{2y}\hat{\sigma}_{3x}|\psi\rangle = |\psi,\rangle \qquad (4)$$

$$\hat{D}|\psi\rangle = \hat{\sigma}_{1x}\hat{\sigma}_{2x}\hat{\sigma}_{3x}|\psi\rangle = -|\psi,\rangle \qquad (5)$$

From equations (2)–(5) we have at once that

$$E(\hat{A}) = E(\hat{B}) = E(\hat{C}) = 1 \qquad (6)$$

and

$$E(ABC) = E(\hat{D}) = -1. \qquad (7)$$

Good reference on above derivation: Mermin, N. D. (1990) *Physical Review Letters*, **65**, 1838.

GHZ Inequalities

$$
\begin{aligned}
-2 &\le E(\mathbf{A}) + E(\mathbf{B}) + E(\mathbf{C}) - E(\mathbf{ABC}) \le 2, \\
-2 &\le -E(\mathbf{A}) + E(\mathbf{B}) + E(\mathbf{C}) + E(\mathbf{ABC}) \le 2, \\
-2 &\le E(\mathbf{A}) - E(\mathbf{B}) + E(\mathbf{C}) + E(\mathbf{ABC}) \le 2, \\
-2 &\le E(\mathbf{A}) + E(\mathbf{B}) - E(\mathbf{C}) + E(\mathbf{ABC}) \le 2.
\end{aligned}
$$

de Barros, J. A. and Suppes, P. (2000) Inequalities for dealing with detector inefficiencies in Greenberger-Horne-Zeilinger-type experiments. *Physical Review Letters*, **84**, 793–797.

Theorem 11 Let $\mathbf{X}_i$ and $\mathbf{Y}_i, 1 \leq i \leq 3$, be six ± 1 random variables such that $E(\mathbf{X}_i) = E(\mathbf{Y}_i) = 0$. Then, there exists a joint probability distribution for all six random variables if and only if the following inequalities are satisfied:

$$-2 \leq E(\mathbf{X}_1\mathbf{Y}_2\mathbf{Y}_3) + E(\mathbf{Y}_1\mathbf{X}_2\mathbf{Y}_3) + E(\mathbf{Y}_1\mathbf{Y}_2\mathbf{X}_3) - E(\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3) \leq 2,$$

$$-2 \leq E(\mathbf{X}_1\mathbf{Y}_2\mathbf{Y}_3) + E(\mathbf{Y}_1\mathbf{X}_2\mathbf{Y}_3) - E(\mathbf{Y}_1\mathbf{Y}_2\mathbf{X}_3) + E(\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3) \leq 2,$$

$$-2 \leq E(\mathbf{X}_1\mathbf{Y}_2\mathbf{Y}_3) - E(\mathbf{Y}_1\mathbf{X}_2\mathbf{Y}_3) + E(\mathbf{Y}_1\mathbf{Y}_2\mathbf{X}_3) + E(\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3) \leq 2,$$

$$-2 \leq -E(\mathbf{X}_1\mathbf{Y}_2\mathbf{Y}_3) + E(\mathbf{Y}_1\mathbf{X}_2\mathbf{Y}_3) + E(\mathbf{Y}_1\mathbf{Y}_2\mathbf{X}_3) + E(\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3) \leq 2.$$

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