Some Results on Generalized Coherence of Conditional Probability Bounds

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Abstract

Based on the coherence principle of de Finetti and a related notion of generalized coherence (g-coherence), we adopt a probabilistic approach to uncertainty based on conditional probability bounds. Our notion of g-coherence is equivalent to the "avoiding uniform loss" property for lower and upper probabilities (a la Walley). Moreover, given a g-coherent imprecise assessment by our algorithms we can correct it obtaining the associated coherent assessment (in the sense of Walley and Williams). As is well known, the problems of checking g-coherence and propagating tight g-coherent intervals are $NP-$ and $FP^{NP}-$complete, respectively, and thus $NP-$hard. Two notions which may be helpful to reduce computational effort are those of non relevant gain and basic set. Exploiting them, our algorithms can use linear systems with reduced sets of variables and/or linear constraints. In this paper we give some insights on the notions of non relevant gain and basic set. We consider several families with three conditional events, obtaining some results characterizing g-coherence in such cases. We also give some more general results.

Keywords

uncertain knowledge, coherence, g-coherence, imprecise probabilities, conditional probability bounds, lower and upper probabilities, non relevant gains, basic sets

1 Introduction

Among the many symbolic or numerical approaches to the management of uncertain knowledge, the probabilistic treatment of uncertainty by means of precise or imprecise assessments is a well known formalism often applied in real situations.

A general framework which allows a consistent management of probabilistic assessments is obtained by resorting to de Finetti's coherence principle ([2], [7], [8], [11]), or suitable generalizations of it given for upper and lower probabilities ([20], [19]). In our approach we adopt the notion of g-coherence (i.e. generalized coherence) introduced in [1] (see also [10]), which is weaker than the notion of coherence given in [19]. Actually, the notion of g-coherence is equivalent to the property of "avoiding uniform loss" given in [19]. Within our framework, a given g-coherent assessment can be corrected, obtaining the associated coherent one, and possibly extended to further conditional events. As is well known, if we discard the case of conditioning events with zero probability the probabilistic reasoning can be reduced to a linear optimization problem (we also point out that g-coherent probabilistic reasoning generally does not coincide with probabilistic reasoning as in, e.g., [12], [14], when the conditioning event has a non-zero probability). When conditioning events may have zero lower/upper probability, the methods presented in the literature (our one too) usually exploit sequences of linear programs. Among them, a "dual" approach for the extension of lower and upper previsions, explicitly based on random gains, has been developed in [20]. With the aim of improving the method given in [20], an interesting technique for computing lower conditional expectations through sequences of pivoting operations has been proposed in [9]. Roughly speaking, probabilistic reasoning can be developed by local approaches, based on the iteration of suitable inference rules, and global ones (the issue of local versus global approaches has been examined especially in [17], [18]). We recall that probabilistic reasoning based on a global approach tends to become intractable. Hence, it is worthwhile to examine any method which try to eliminate or reduce computational difficulties, possibly finding efficient special-case algorithms. This problem has been faced by many authors (see, e.g., [5], [7], [8], [9], [12], [14], [20]). Many aspects concerning the complexity of probabilistic reasoning under coherence have been studied in [3]. The relationship between coherence-based and model-theoretic probabilistic reasoning has been widely explored in [4]. In [16] an efficient procedure has been proposed for families of *conjunctive* conditional events. Such procedure can be characterized in the framework of coherence introducing suitable notions of non relevant gains and basic sets ([2]). Exploiting such notions, our algorithms for g-coherence checking and propagation of conditional probability bounds can use linear systems with reduced sets of variables and/or constraints. In this paper we illustrate the notions of non relevant gain and basic set, by examining several examples of families constituted by three conditional events. We obtain some theoretical results which characterize g-coherence in such particular cases. In this way, the characterization of g-coherence in the case of larger families of conditional events should be facilitated. We obtain some necessary and sufficient conditions for the g-coherence of lower probability bounds. We also give some more general results. Notice that the case of families with three conditional events may have a specific importance, e.g., in the field of default reasoning where many inference rules consist of two premises and one consequence. We also recall that coherence-based probabilistic reasoning can be reduced to standard reasoning tasks in model-theoretic probabilistic logic, using concepts from default reasoning ([4]). The rest of the paper is organized as follows. In Section 2 we recall some preliminary concepts. In Section 3 we illustrate the notions of non relevant gain and basic set and we recall some theoretical results. In Section 4 we consider several cases of families constituted by three conditional events and we give some necessary and sufficient conditions of g-coherence. In Section 5 we give some more general results. Finally, in Section 6 we give some conclusions and an outlook on further developments.

2 Some preliminary concepts

For each integer n, we set $J_n = \{1,\ldots,n\}$. Given any event E, we denote by the same symbol its indicator and by E^c its negation. Given a further event H, we denote by EH (resp. $E \vee H$) the conjunction (resp. disjunction) of E and H. Let P be a conditional probability assessment defined on a family of conditional events $\mathcal{K}$. Given a finite subfamily $\mathcal{F}_n = \{E_1|H_1,\ldots,E_n|H_n\} \subseteq \mathcal{K}$, let $\mathcal{P}_n$ be the vector $(p_1,\ldots,p_n)$, where $p_i = P(E_i|H_i), i \in J_n$. With the pair $(\mathcal{F}_n,\mathcal{P}_n)$ we associate the random quantity $G_n = \sum_{i\in J_n} s_i H_i(E_i - p_i)$, with $s_1,\ldots,s_n$ arbitrary real numbers. Moreover, we denote by $G_n|\mathcal{H}_n$ the restriction of G_n to $\mathcal{H}_n = H_1 \vee \cdots \vee H_n$. Then, based on the *betting scheme*, we have

Definition 1 The probability assessment P on $\mathcal{K}$ is said coherent if, for every integer $n = 1,2,\ldots$, for every subfamily $\mathcal{F}_n \subseteq \mathcal{K}$ and for every real numbers $s_1,\ldots,s_n$, the condition $Max\ G_n|\mathcal{H}_n \geq 0$ is satisfied.

We denote by $\mathcal{A}_n$ a vector $(\alpha_1,\ldots,\alpha_n)$ of lower probability bounds on $\mathcal{F}_n$. We say that the pair $(\mathcal{F}_n,\mathcal{A}_n)$ is associated with the set J_n.

Definition 2 The vector of lower bounds $\mathcal{A}_n$ is g-coherent iff there exists a coherent probability assessment $\mathcal{P}_n = (p_1,\ldots,p_n)$ on $\mathcal{F}_n$ such that $p_i \geq \alpha_i, \forall i \in J_n$.

By expanding the expression $\bigwedge_{i\in J_n}(E_iH_i \vee E_i^cH_i \vee H_i^c)$, we obtain the constituents associated with $\mathcal{F}_n$. We denote by $C_1,\ldots,C_m$, where $m \leq 3^n - 1$, the constituents contained in $\mathcal{H}_n = \bigvee_{j\in J_n} H_j$. A further constituent (if it is not impossible) is $C_0 = \mathcal{H}_n^c = H_1^c \cdots H_n^c$.

Remark: With the family $\mathcal{F}_n$ we associate a set L which describe the logical relationships among the events $E_i, H_i, i \in J_n$. Then, the set of constituents is the set of those conjunctions $\chi_1 \cdots \chi_n$, with $\chi_i \in \{E_iH_i, E_i^cH_i, H_i^c\}, \forall i \in J_n$, which satisfy the set of logical relations L. Notice that, if $L = \emptyset$, then $m = 3^n - 1$ and $C_0 \neq \emptyset$, i.e. the number of constituents is 3^n.

For each constituent $C_r, r \in J_m$, we introduce a vector $V_r = (v_{r1},\ldots,v_{rn})$, where for each $i \in J_n$ it is respectively $v_{ri} = 1$, or $v_{ri} = 0$, or $v_{ri} = \alpha_i$, according to whether $C_r \subseteq E_i H_i$, or $C_r \subseteq E_i^c H_i$, or $C_r \subseteq H_i^c$. With the pair $(\mathcal{F}_n, \mathcal{A}_n)$ we associate the random gain $G_n = \sum_{i \in J_n} s_i H_i (E_i - \alpha_i)$, where $s_i \geq 0$, $\forall\, i \in J_n$. Moreover, we denote by

$$g_h = G_n(V_h) = \sum_{i \in J_n} s_i (v_{hi} - \alpha_i) = \sum_{i : C_h \subseteq H_i} s_i (v_{hi} - \alpha_i) \tag{1}$$

the value of $G_n | \mathcal{H}_n$ associated with C_h. We denote by $(\mathcal{S}_n)$ the following system in the unknowns λ_r's.

$$\sum_{r \in J_m} \lambda_r v_{ri} \geq \alpha_i, \;\; i \in J_n; \qquad \sum_{r \in J_m} \lambda_r = 1; \qquad \lambda_r \geq 0, \;\; \forall\, r \in J_m. \tag{2}$$

Remark: The solvability of $(\mathcal{S}_n)$ means that there exists a non negative vector $(\lambda_r \,;\; r \in J_m)$, with $\sum_{r \in J_m} \lambda_r = 1$, such that $\sum_{r \in J_m} \lambda_r V_r \geq \mathcal{A}_n$. In other words, in the convex hull of the points V_r's there exists a point $V^* = \sum_{r \in J_m} \lambda_r V_r$ such that $V^* \geq \mathcal{A}_n$ (this geometrical approach will be used in the proof of Theorem 4).

As shown in [10], a set of lower bounds $\mathcal{A}$ defined on $\mathcal{K}$ is g-coherent iff, for every n and for every $\mathcal{F}_n \subseteq \mathcal{K}$, the system (2) is solvable. Moreover, based on a suitable alternative theorem, it can be shown ([2]) that the solvability of system (2) is equivalent to the following condition

$$Max\ G_n | \mathcal{H}_n \geq 0. \tag{3}$$

Then, we have

Proposition 1 A set of lower bounds $\mathcal{A}$ defined on a family of conditional events $\mathcal{K}$ is g-coherent iff $\forall\, n, \forall\, \mathcal{F}_n \subseteq \mathcal{K}$, and $\forall\, s_i \geq 0\,, i \in J_n$, it is $Max\ G_n | \mathcal{H}_n \geq 0$.

We remark that, if the case of zero probability for conditioning events is discarded, then to check g-coherence of the assessment $\mathcal{A}_n$ on $\mathcal{F}_n$ it is enough to check solvability of system (2). However, in our coherence-based approach, some (or possibly all) conditioning events may have zero probability. Then, to check g-coherence we should study the solvability of a very large number of systems, like (2). Actually, we can exploit algorithms which only check (the solvability of) a small number of linear systems (see, e.g., [1], [2], [5]).

3 Non relevant gains and basic sets

In this section we illustrate the notions of non relevant gain and basic set. Exploiting such notions, the algorithms for g-coherence checking and propagation of conditional probability bounds can use linear systems with reduced sets of variables and/or constraints. We recall some theoretical conditions given in [2].

Definition 3 Let $\mathcal{G} = \{g_j\}_{j \in J_m}$ be the set of possible values of the random gain $G_n | \mathcal{H}_n$. Then, a value $g_r \in \mathcal{G}$ is said *"not relevant for the checking of condition (3)"*, or in short *"not relevant"*, if there exists a set $T_r \subseteq J_m \setminus \{r\}$ such that:

$$Max\, \{g_j\}_{j \in T_r} < 0 \implies g_r < 0\,. \tag{4}$$

Remark: Notice that, in the previous definition, it wouldn't be equivalent to use the condition $T_r = J_m \setminus \{r\}$ instead of $T_r \subseteq J_m \setminus \{r\}$. In fact, it may happen that (4) holds with $T_r \subset J_m \setminus \{r\}$, so that g_r is not relevant, while at the same time it may be $Max\,\{g_j\}_{j\in J_m\setminus\{r\}} > 0$.

Definition 4 A set $\mathcal{G}_\Gamma = \{g_r\}_{r\in\Gamma}$, with $\Gamma \subset J_m$, is said *not relevant* if, $\forall r \in \Gamma$, there exists a set $T_r \subseteq J_m \setminus \Gamma$ such that (4) is satisfied.

Definition 5 A set $\mathcal{T} \subset J_m$ is said *basic* if the following property holds:
Basic Property. For every $r \in J_m \setminus \mathcal{T}$ there exists a set $T_r \subseteq \mathcal{T}$ such that the condition (4) is satisfied.
A basic set $\mathcal{T}$ is said *minimal* if, for every $T \subset \mathcal{T}$, the set T is not basic.

We observe that $Max\,G_n|\mathcal{H}_n \;=\; Max\,\{g_j\}_{j\in J_m}$. Then, we have

Theorem 1 Let $\mathcal{T} \subset J_m$ be a *basic* set. Then

$$Max\,\{g_j\}_{j\in J_m} \;\geq\; 0 \quad\Longleftrightarrow\quad Max\,\{g_j\}_{j\in\mathcal{T}} \;\geq\; 0\,. \tag{5}$$

Remark: We point out that, given a subset $\mathcal{T}$, if there exists $r \notin \mathcal{T}$ such that, for every $T_r \subseteq \mathcal{T}$, the condition (3) is not satisfied, then $\mathcal{T}$ is not a basic set. Moreover, we observe that the condition (5) is trivially satisfied for $\mathcal{T} = J_m$. Then, as for $\mathcal{T} = J_m$ the set $J_m \setminus \mathcal{T}$ is empty, we can enlarge the class of basic sets by including in it J_m too.

Given $r \in J_m$ and a set $\mathcal{T}_r \subseteq J_m \setminus \{r\}$, let us consider the following condition

$$g_r \leq \sum_{j\in\mathcal{T}_r} a_j g_j\,; \quad a_j > 0\,,\, \forall j \in \mathcal{T}_r\,. \tag{6}$$

By Definition 3 one has that, if the above condition is satisfied, then g_r is not relevant. The condition (6) can be exploited in general to reduce the number of variables. The basic idea is illustrated by the following theorem ([2], [5]).

Theorem 2 Let $\mathcal{T}$ be a strict subset of the set J_m such that for every $r \notin \mathcal{T}$ there exists $T_r \subseteq \mathcal{T}$ satisfying the condition (6). Then:

$$Max\,\{g_j\}_{j\in J_m} \geq 0 \Longleftrightarrow Max\,\{g_j\}_{j\in\mathcal{T}} \geq 0\,. \tag{7}$$

Based on the previous result and on suitable alternative theorems, in order to check g-coherence we can replace $(\mathcal{S}_n)$ by an equivalent system $(\mathcal{S}_n^{\mathcal{T}})$, which has a reduced vector of unknowns $\Lambda_T = (\lambda_r; r \in \mathcal{T})$. We denote by $S_{\mathcal{T}}$ the set of solutions of $(\mathcal{S}_n^{\mathcal{T}})$. Moreover, for each $j \in J_n$, we consider the function $\Phi_j^{\mathcal{T}}(\Lambda_T) = \sum_{r\in\mathcal{T}:C_r\subseteq H_j} \lambda_r$. We denote by $I_0^{\mathcal{T}}$ the (strict) subset of J_n defined as

$$I_0^{\mathcal{T}} = \{j \in J_n \;:\; M_j = Max_{\Lambda_{\mathcal{T}}\in S_{\mathcal{T}}} \Phi_j^{\mathcal{T}}(\Lambda_{\mathcal{T}}) = 0\} \tag{8}$$

and by $(\mathcal{F}_0^{\mathcal{T}}, \mathcal{A}_0^{\mathcal{T}})$ the pair associated with $I_0^{\mathcal{T}}$. Then, to check g-coherence of $\mathcal{A}_n$, we can exploit the following result ([2]).

Theorem 3 The imprecise assessment $\mathcal{A}_n$ on $\mathcal{F}_n$ is g-coherent if and only if: 1) the system $(\mathcal{S}_n^{\mathcal{T}})$ is solvable; 2) if $I_0^{\mathcal{T}} \neq \emptyset$, then $\mathcal{A}_0^{\mathcal{T}}$ is g-coherent.

Note that, if $|I_0^{\mathcal{T}}| = 1$, say $I_0^{\mathcal{T}} = \{h\}$, then $\mathcal{A}_0^{\mathcal{T}} = (\alpha_h)$ and the g-coherence of $\mathcal{A}_0^{\mathcal{T}}$ simply amounts to the condition: $\alpha_h \leq 1$.

4 Some results on g-coherence of lower probability bounds for families of three conditional events

In this section we will illustrate the notions of non relevant gain and basic set by examining several examples which concern particular families of three conditional events.

Remark: We recall that such kind of families may be relevant in the field of default reasoning, where many inference rules are associated with two premises and one conclusion. As an example, with the following basic inference rules of System *P* ([15])

$$A \mid\!\sim B,\, A \mid\!\sim C \implies A \mid\!\sim BC\,, \qquad (And),$$

$$A \mid\!\sim C,\, A \mid\!\sim B \implies AB \mid\!\sim C\,, \qquad (Cautious\,Monotonicity),$$

$$A \mid\!\sim C,\, B \mid\!\sim C \implies A \vee B \mid\!\sim C\,, \qquad (Or),$$

are associated, respectively, the following families of conditional events

$$\{B|A,\ C|A, BC|A\}\,; \quad \{C|A, B|A, C|AB\}\,; \quad \{C|A, C|B, C|(A \vee B)\}\,.$$

We also note that the theoretical results obtained in the case $n = 3$ may be useful in establishing more general results when $n > 3$.

In what follows, to avoid the analysis of trivial or particular cases, we assume

$$\emptyset \subset E_iH_i \subset H_i\,, \quad 0 < \alpha_i < 1\,, \quad \forall\, i\,.$$

Then, for each $r \in J_m$, as $\alpha_i < 1$, if $v_{ri} = 1$ for some i, it follows $C_r \subseteq E_iH_i$.
Let $\mathcal{A}_3 = (\alpha_1, \alpha_2, \alpha_3)$ be a vector of lower bounds on $\mathcal{F}_3 = \{E_1|H_1, E_2|H_2, E_3|H_3\}$.
Given the set $\mathcal{V} = \{V_1, \ldots, V_m\}$, we define

$$\mathcal{W} = \{V_r \in \mathcal{V} : v_{ri} \neq 0, \forall\, i \in J_n\} \tag{9}$$

and, for each $V_r \in \mathcal{W}$,

$$N_r = \{i \in J_n : C_r \subseteq H_i^c\}\,. \tag{10}$$

Of course, $N_r \subset J_n$. Then, we define

$$\mathcal{V}_h = \{V_r \in \mathcal{W} : |N_r| = h\},\ h = 0, 1, \ldots, n-1\,. \tag{11}$$

With each $V_r \in \mathcal{V}$, $r \in J_m$, we associate the set N_r defined in (10) and the set

$$M_r = \{i \in J_n : v_{ri} = 0\}\,. \tag{12}$$

Then, introducing the set $I = \{(h,k) : h = 0,\ldots,n-1;\ k = 1,\ldots,n\}$, we define the sets

$$\mathcal{U}_{h,k} = \{V_r \in \mathcal{V} : |N_r| = h\,,\ |M_r| = k\}, \quad (h,k) \in I\,. \tag{13}$$

We observe that, if the sets $\mathcal{U}_{h,0}$ were defined, then recalling (11) we would have $\mathcal{V}_h = \mathcal{U}_{h,0}$. Then, recalling (9), we have

$$\mathcal{V} = \mathcal{W} \cup (\bigcup_{(h,k)\in I} \mathcal{U}_{h,k}) = (\bigcup_{h=0}^{n-1} \mathcal{V}_h) \cup (\bigcup_{h,k} \mathcal{U}_{h,k})\,. \tag{14}$$

As $n = 3$, the set of vectors $\mathcal{V} = \{V_1,\ldots,V_m\}$, where $m \le 26$, is a subset of the set

$$\{(1,1,1),(1,1,\alpha_3),(1,\alpha_2,1),(\alpha_1,1,1),\ldots,(\alpha_1,0,0),(0,\alpha_2,0),(0,0,\alpha_3),(0,0,0)\}\,.$$

By (14), we have

$$\mathcal{V} = \mathcal{V}_0 \cup \mathcal{V}_1 \cup \mathcal{V}_2 \cup \mathcal{U}_{0,1} \cup \mathcal{U}_{1,1} \cup \mathcal{U}_{0,2} \cup \mathcal{U}_{2,1} \cup \mathcal{U}_{1,2} \cup \mathcal{U}_{0,3}\,, \tag{15}$$

where

$$\mathcal{V}_0 \subseteq \{(1,1,1)\}\,, \quad \mathcal{V}_1 \subseteq \{(1,1,\alpha_3),(1,\alpha_2,1),(\alpha_1,1,1)\}\,,$$

$$\mathcal{V}_2 \subseteq \{(1,\alpha_2,\alpha_3),(\alpha_1,1,\alpha_3),(\alpha_1,\alpha_2,1)\}\,, \quad \mathcal{U}_{0,1} \subseteq \{(1,1,0),(1,0,1),(0,1,1)\}\,,$$

$$\mathcal{U}_{1,1} \subseteq \{(1,\alpha_2,0),(1,0,\alpha_3),(\alpha_1,1,0),(0,1,\alpha_3),(\alpha_1,0,1),(0,\alpha_2,1),\}\,,$$

$$\mathcal{U}_{0,2} \subseteq \{(1,0,0),(0,1,0),(0,0,1)\}\,, \quad \mathcal{U}_{2,1} \subseteq \{(\alpha_1,\alpha_2,0),(\alpha_1,0,\alpha_3),(0,\alpha_2,\alpha_3)\}\,,$$

$$\mathcal{U}_{1,2} \subseteq \{(\alpha_1,0,0),(0,\alpha_2,0),(0,0,\alpha_3)\}\,, \quad \mathcal{U}_{0,3} \subseteq \{(0,0,0)\}\,.$$

Remark: Notice that each given set of logical relationships L among the events $E_i, H_i, i = 1,2,3$, determines a particular representation (15) for the set of vectors $\mathcal{V}$. Then, in what follows, instead of assigning the set L, we directly assume some hypotheses on the subsets $\mathcal{V}_h$'s and $\mathcal{U}_{h,k}$'s. We list below some sufficient conditions, proved in [6], for g-coherence of the vector of lower bounds $\mathcal{A}_3$ on $\mathcal{F}_3$.
1. $|\mathcal{V}_0| = 1$; 2. $\mathcal{V}_0 = \emptyset,\ |\mathcal{V}_1| \ge 1$; 3. $\mathcal{V}_0 = \mathcal{V}_1 = \emptyset, |\mathcal{V}_2| \ge 2$;
4. $\mathcal{V}_0 = \mathcal{V}_1 = \emptyset,\ \mathcal{V}_2 = \{(1,\alpha_2,\alpha_3)\},\ E_2H_2E_3H_3 \vee E_2H_2H_3^c \vee H_2^cE_3H_3 \neq \emptyset$.
Some further conditions obtained in [6] are given below.
5. If $\mathcal{V}_0 = \mathcal{V}_1 = \emptyset,\ \mathcal{V}_2 = \{(1,\alpha_2,\alpha_3)\},\ E_2H_2E_3H_3 = E_2H_2H_3^c = H_2^cE_3H_3 = \emptyset$, then $\mathcal{A}_3$ is g-coherent iff $\alpha_2 + \alpha_3 \le 1$.
6. $\mathcal{V}_0 = \mathcal{V}_1 = \mathcal{V}_2 = \emptyset,\ \alpha_1 + \alpha_2 + \alpha_3 > 2 \implies \mathcal{A}_3$ not g-coherent.

7. If $\mathcal{V}_0 = \mathcal{V}_1 = \mathcal{V}_2 = \emptyset$, $|\mathcal{U}_{0,1}| = 3$, $\alpha_i < 1, \forall i$, then: a) there exists a basic set $\mathcal{T}$, with $|\mathcal{T}| = 3$; b) $\mathcal{A}_3$ is g-coherent iff $\alpha_1 + \alpha_2 + \alpha_3 \leq 2$.
8. If $\mathcal{V}_0 = \mathcal{V}_1 = \mathcal{V}_2 = \mathcal{U}_{0,1} = \mathcal{U}_{1,1} = \emptyset$, $\mathcal{U}_{0,2} = \{(1,0,0),(0,1,0),(0,0,1))\}$, $\alpha_i < 1, \forall i$, then:
a) if $\alpha_1 + \alpha_2 \leq 1, \alpha_1 + \alpha_3 \leq 1, \alpha_2 + \alpha_3 \leq 1$, then $\mathcal{T} = \{1,2,3\}$ is a basic set;
b) $\mathcal{A}_3$ is g-coherent iff $\alpha_1 + \alpha_2 + \alpha_3 \leq 1$.

Now we give further results concerning the case $n = 3$. Besides providing a better understanding of the notions of basic set and non relevant gain, these results permit in particular the deepening of the condition (6). In next theorem the hypotheses concerning the set of logical relations L specify that the conjunctions

$$\begin{array}{llll} E_1H_1E_2H_2E_3H_3\,, & E_1H_1E_2H_2H_3^c\,, & E_1H_1H_2^cE_3H_3\,, & H_1^cE_2H_2E_3H_3\,, \\ E_1H_1H_2^cH_3^c\,, & H_1^cE_2H_2H_3^c\,, & H_1^cH_2^cE_3H_3\,, & E_1^cH_1E_2H_2E_3H_3 \end{array}$$

are impossible, while the conjunctions

$$E_1H_1E_2H_2E_3^cH_3\,, \quad E_1H_1E_2^cH_2E_3H_3\,, \quad E_1^cH_1E_2H_2H_3^c\,, \quad E_1^cH_1H_2^cE_3H_3$$

are possible. Then, concerning the number m of unknowns in the system $(\mathcal{S}_3)$, one has: $4 \leq m \leq 18$. Actually, we will use a system $(\mathcal{S}_3^{\mathcal{T}})$ with only 3 or 4 unknowns.

Theorem 4 If $\mathcal{V}_0 = \mathcal{V}_1 = \mathcal{V}_2 = \emptyset$, $\mathcal{U}_{0,1} = \{V_1, V_2\} = \{(1,1,0),(1,0,1)\}, \{V_3, V_4\} = \{(0,1,\alpha_3),(0,\alpha_2,1)\} \subseteq \mathcal{U}_{1,1}$, $0 < \alpha_i < 1, \forall i$, then one has:
a) for every $r > 4$, the gain g_r is not relevant;
b) if $\alpha_1 + \alpha_2 \leq 1$, or $\alpha_2 + \alpha_3 \leq 1$, or $\alpha_1 + \alpha_3 \leq 1$, then there exists a basic set $\mathcal{T}$, with $|\mathcal{T}| \leq 3$, and $\mathcal{A}_3$ is g-coherent;
c) if $\alpha_1 + \alpha_2 > 1, \alpha_2 + \alpha_3 > 1, \alpha_1 + \alpha_3 > 1$, then $\mathcal{A}_3$ is g-coherent iff

$$\alpha_1\alpha_3 + \alpha_2 \leq 1\,, \quad \text{or} \quad \alpha_1\alpha_2 + \alpha_3 \leq 1.$$

Proof. a) by the hypotheses, it follows that for each $V_r \in \mathcal{V}$, with $r > 4$, there exists $h \in \{1,2,3,4\}$ such that $V_r \leq V_h$; hence g_r is not relevant. Then, $\mathcal{T} = \{1,2,3,4\}$ is a basic set.
In order to study the g-coherence of $\mathcal{A}_3$, we first determine the gains associated with the vectors V_1, V_2, V_3, V_4. Recalling (1), these gains are respectively

$$\begin{array}{ll} g_1 = s_1(1-\alpha_1) + s_2(1-\alpha_2) - s_3\alpha_3\,, & g_2 = s_1(1-\alpha_1) - s_2\alpha_2 + s_3(1-\alpha_3)\,, \\ g_3 = -s_1\alpha_1 + s_2(1-\alpha_2)\,, & g_4 = -s_1\alpha_1 + s_3(1-\alpha_3)\,. \end{array}$$

We also need the equations of the planes $\pi_1, \pi_2, \pi_3, \pi_4$, containing respectively the triangles $V_1V_2V_3$, $V_1V_2V_4$, $V_1V_3V_4$, $V_2V_3V_4$, which are given below

$$\begin{array}{ll} \pi_1: \ \alpha_3 x + y + z = 1 + \alpha_3\,; & \pi_3: \ \alpha_3(1-\alpha_2)x + (1-\alpha_3)y + (1-\alpha_2)z = 1 - \alpha_2\alpha_3\,; \\ \pi_2: \ \alpha_2 x + y + z = 1 + \alpha_2\,; & \pi_4: \ \alpha_2(1-\alpha_3)x + (1-\alpha_3)y + (1-\alpha_2)z = 1 - \alpha_2\alpha_3\,. \end{array}$$

The intersection points of the segment $(x,\alpha_2,\alpha_3),\ 0 \le x \le 1,$ with the planes π_1 and π_2, are respectively $V_x^* = (\frac{1-\alpha_2}{\alpha_3},\alpha_2,\alpha_3)$ and $V_x^{**} = (\frac{1-\alpha_3}{\alpha_2},\alpha_2,\alpha_3)$. Moreover,

$$V_x^* \ge \mathcal{A}_3 \iff \alpha_1\alpha_3+\alpha_2 \le 1\,; \qquad V_x^{**} \ge \mathcal{A}_3 \iff \alpha_1\alpha_2+\alpha_3 \le 1\,.$$

The intersection point of the segment $(\alpha_1,y,\alpha_3),\ 0 \le y \le 1,$ with the plane π_3 is

$$V_y^* = (\alpha_1, \tfrac{1-\alpha_3-\alpha_1\alpha_3(1-\alpha_2)}{1-\alpha_3}, \alpha_3) \ge \mathcal{A}_3\,, \qquad \forall\, \alpha_2 \in [0,1]\,.$$

The intersection point of the segment $(\alpha_1,\alpha_2,z),\ 0 \le z \le 1,$ with the plane π_4 is

$$V_z^* = (\alpha_1,\alpha_2, \tfrac{1-\alpha_2-\alpha_1\alpha_2(1-\alpha_3)}{1-\alpha_2}) \ge \mathcal{A}_3\,, \qquad \forall\, \alpha_3 \in [0,1]\,.$$

b.1) assume that $\alpha_1+\alpha_2 \le 1$ and consider the set

$$S = \{(a,b)\,:\, a \ge \frac{1-\alpha_2}{1-\alpha_1-\alpha_2}\,,\ 1+\frac{\alpha_2}{1-\alpha_2}a \le b \le \frac{1-\alpha_1}{\alpha_1}a - \frac{1-\alpha_1}{\alpha_1}\}\,.$$

We have: $a>0\,,\ b>0\,,\ ag_2+bg_3 \ge g_1\,,\ \forall (a,b) \in S$. Then, g_1 is not relevant and $\mathcal{T}=\{2,3,4\}$ is a basic set. Moreover, $V_z^* = \lambda_2V_2+\lambda_3V_3+\lambda_4V_4$, with

$$\lambda_2 = \alpha_1\,,\quad \lambda_3 = \frac{\alpha_1\alpha_2}{1-\alpha_2}\,,\quad \lambda_4 = \frac{1-\alpha_1-\alpha_2}{1-\alpha_2}\,.$$

We recall that $0<\alpha_i<1\,,\ i=1,2,3$, so that $\lambda_2>0\,,\ \lambda_3>0\,,\ \lambda_4 \ge 0$. Then, the vector $(\lambda_2,\lambda_3,\lambda_4)$ is a solution of the system $(\mathcal{S}_3^{\mathcal{T}})$, with $|I_0^{\mathcal{T}}| \le 1$, and hence, by Theorem 3, $\mathcal{A}_3$ is g-coherent.

b.2) assume that $\alpha_2+\alpha_3 \le 1$ and consider the sets

$$S_1 = \{(a,b)\,:\, 0<a \le \frac{1-\alpha_2-\alpha_3+\alpha_2\alpha_3}{1-\alpha_2-\alpha_3}\,,\ \frac{\alpha_3}{1-\alpha_3}a \le b \le \frac{1-\alpha_2}{\alpha_2}a - \frac{1-\alpha_2}{\alpha_2}\}\,;$$

$$S_2 = \{(\gamma,\delta)\,:\, 0<\gamma \le \frac{\alpha_2\alpha_3(1-\alpha_3)}{1-\alpha_2-\alpha_3}\,,\ 1+\frac{\alpha_3}{1-\alpha_3}\gamma \le \delta \le \frac{1-\alpha_2}{\alpha_2}\gamma\}\,.$$

For each $(a,b) \in S_1$, $(\gamma,\delta) \in S_2$, one has

$$a>0\,,\quad b>0\,,\quad \gamma>0\,,\quad \delta>0\,,\quad ag_1+bg_2 \ge g_3\,,\quad \gamma g_1+\delta g_2 \ge g_4\,.$$

Then, g_3 and g_4 are not relevant and $\mathcal{T}=\{1,2\}$ is a basic set. Moreover, defining $V^* = (1,\alpha_2,1-\alpha_2)$, $\lambda_1=\alpha_2$, $\lambda_2 = 1-\alpha_2$, one has

$$V^* \ge (1,\alpha_2,\alpha_3) \ge \mathcal{A}_3\,; \quad V^* = \lambda_1V_1+\lambda_2V_2\,,\ \lambda_1>0\,,\ \lambda_2>0\,,\ \lambda_1+\lambda_2=1\,.$$

Then, the vector (λ_1,λ_2) is a solution of the system $(\mathcal{S}_3^{\mathcal{T}})$, with $I_0^{\mathcal{T}}=\emptyset$, and hence, by Theorem 3, $\mathcal{A}_3$ is g-coherent.

b.3) assume that $\alpha_1+\alpha_3 \le 1$ and consider the set

$$S = \{(a,b)\,:\, a \ge \frac{1-\alpha_3}{1-\alpha_1-\alpha_3}\,,\ 1+\frac{\alpha_3}{1-\alpha_3}a \le b \le \frac{1-\alpha_1}{\alpha_1}a - \frac{1-\alpha_1}{\alpha_1}\}\,.$$

We have: $a > 0\,,\ b > 0\,,\ \ ag_1 + bg_4 \ \geq\ g_2\,,\ \ \forall (a,b) \in S$. Then, g_2 is not relevant and $\mathcal{T} = \{1,3,4\}$ is a basic set. Moreover, $V_y^* = \lambda_1 V_1 + \lambda_3 V_3 + \lambda_4 V_4$, with

$$\lambda_1 = \alpha_1 > 0\,,\quad \lambda_3 = \frac{1-(\alpha_1+\alpha_3)}{1-\alpha_3} \geq 0\,,\quad \lambda_4 = \frac{\alpha_1\alpha_3}{1-\alpha_3} > 0\,.$$

Then, the vector $(\lambda_1, \lambda_3, \lambda_4)$ is a solution of the system $(\mathcal{S}_3^{\mathcal{T}})$, with $|I_0^{\mathcal{T}}| \leq 1$, and hence, by Theorem 3, $\mathcal{A}_3$ is g-coherent. Therefore, under the condition

$$\alpha_1 + \alpha_2 \leq 1\,,\quad \text{or}\quad \alpha_2 + \alpha_3 \leq 1\,,\quad \text{or}\quad \alpha_1 + \alpha_3 \leq 1\,,$$

$\mathcal{A}_3$ is g-coherent.

c) assume that $\alpha_1 + \alpha_2 > 1$, $\alpha_2 + \alpha_3 > 1$, $\alpha_1 + \alpha_3 > 1$.

c.1) if $\alpha_1\alpha_3 + \alpha_2 \leq 1$, then $V_x^* \geq \mathcal{A}_3$. Moreover, $V_x^* = \lambda_1 V_1 + \lambda_2 V_2 + \lambda_3 V_3$, with

$$\lambda_1 = \frac{(1-\alpha_2)(1-\alpha_3)}{\alpha_3} > 0\,,\quad \lambda_2 = 1-\alpha_2 > 0\,,\quad \lambda_3 = \frac{\alpha_2+\alpha_3-1}{\alpha_3} > 0\,.$$

Then, considering the basic set $\mathcal{T} = \{1,2,3,4\}$, the vector $(\lambda_1, \lambda_2, \lambda_3, 0)$ is a solution of the system $(\mathcal{S}_3^{\mathcal{T}})$, with $I_0^{\mathcal{T}} = \emptyset$, and hence, by Theorem 3, $\mathcal{A}_3$ is g-coherent.

c.2) if $\alpha_1\alpha_2 + \alpha_3 \leq 1$, then $V_x^{**} \geq \mathcal{A}_3$. Moreover, $V_x^{**} = \lambda_1 V_1 + \lambda_2 V_2 + \lambda_4 V_4$, with

$$\lambda_1 = 1-\alpha_3 > 0\,,\quad \lambda_2 = \frac{(1-\alpha_2)(1-\alpha_3)}{\alpha_2} > 0\,,\quad \lambda_4 = \frac{\alpha_2+\alpha_3-1}{\alpha_2} > 0\,.$$

Then, considering the basic set $\mathcal{T} = \{1,2,3,4\}$, the vector $(\lambda_1, \lambda_2, 0, \lambda_4)$ is a solution of the system $(\mathcal{S}_3^{\mathcal{T}})$, with $I_0^{\mathcal{T}} = \emptyset$, and hence, by Theorem 3, $\mathcal{A}_3$ is g-coherent.

c.3) assume that $\alpha_1\alpha_2 + \alpha_3 > 1$, $\alpha_1\alpha_3 + \alpha_2 > 1$, and let us make the (absurd) hypothesis that $\mathcal{A}_3$ were g-coherent. Then, considering the basic set $\mathcal{T} = \{1,2,3,4\}$, the system $(\mathcal{S}_3^{\mathcal{T}})$ should be solvable and hence, for suitable non negative values $\lambda_1, \ldots, \lambda_4$, with $\lambda_1 + \cdots + \lambda_4 = 1$, defining

$$V^* = \lambda_1 V_1 + \lambda_2 V_2 + \lambda_3 V_3 + \lambda_4 V_4 = (\lambda_1 + \lambda_2\,, \lambda_1 + \lambda_3 + \alpha_2\lambda_4\,, \lambda_2 + \alpha_3\lambda_3 + \lambda_4)\,,$$

it should be: $V^* \geq \mathcal{A}_3$, that is

$$\lambda_1 + \lambda_2 \geq \alpha_1\,;\quad \lambda_1 + \lambda_3 \geq \alpha_2 - \alpha_2\lambda_4\,;\quad \lambda_2 + \lambda_4 \geq \alpha_3(\lambda_1 + \lambda_2 + \lambda_4)\,, \tag{16}$$

or, equivalently

$$\lambda_1 + \lambda_2 \geq \alpha_1\,;\quad \lambda_1 + \lambda_3 \geq \alpha_2(\lambda_1 + \lambda_2 + \lambda_3)\,;\quad \lambda_2 + \lambda_4 \geq \alpha_3 - \alpha_3\lambda_3\,. \tag{17}$$

Then, assuming $\alpha_3 - \alpha_2 \geq 0$ and recalling that $\alpha_1\alpha_3 + \alpha_2 > 1$, by summing the last two inequalities in (16) we would obtain

$$1 \geq \alpha_3(\lambda_1 + \lambda_2) + \alpha_2 + (\alpha_3 - \alpha_2)\lambda_4 \geq \alpha_1\alpha_3 + \alpha_2 + (\alpha_3 - \alpha_2)\lambda_4 > 1\,,$$

which is absurd. On the other hand, assuming $\alpha_3 - \alpha_2 < 0$ and recalling that $\alpha_1\alpha_2 + \alpha_3 > 1$, by summing the last two inequalities in (17) we would obtain

$$1 \geq \alpha_2(\lambda_1 + \lambda_2) + \alpha_3 + (\alpha_2 - \alpha_3)\lambda_3 \geq \alpha_1\alpha_2 + \alpha_3 + (\alpha_2 - \alpha_3)\lambda_3 > 1\,,$$

which is absurd too. Hence, $(\mathcal{S}_3^{\mathcal{T}})$ is not solvable and $\mathcal{A}_3$ is not g-coherent. □

We observe that the hypotheses concerning the set of logical relations L can be modified in many ways. Then, by the same reasoning as in Theorem 4, we obtain many similar results, which we give without proof in the remaining part of this section (the proofs of these results can be found in [6]).

Theorem 5 If $\mathcal{V}_0 = \mathcal{V}_1 = \mathcal{V}_2 = \emptyset,\ \mathcal{U}_{0,1} = \{V_1, V_2\} = \{(1,1,0),(1,0,1)\}, V_3 = (0,1,\alpha_3) \in \mathcal{U}_{1,1}, (0,\alpha_2,1) \notin \mathcal{U}_{1,1},\ \alpha_i < 1\,, \forall i$, then one has:
a) for every $r > 3$, the gain g_r is not relevant;
b) if $\alpha_2 + \alpha_3 \leq 1$, then there exists a basic set $\mathcal{T}$, with $|\mathcal{T}| = 2$, and $\mathcal{A}_3$ is g-coherent;
c) if $\alpha_2 + \alpha_3 > 1$, then $\mathcal{A}_3$ is g-coherent iff $\alpha_1\alpha_3 + \alpha_2 \leq 1$.

Theorem 6 If $\mathcal{V}_0 = \mathcal{V}_1 = \mathcal{V}_2 = \emptyset,\ \mathcal{U}_{0,1} = \{V_1\} = \{(1,1,0)\}, \{V_2, V_3, V_4, V_5\} = \{(\alpha_1,0,1),(0,\alpha_2,1),(1,0,\alpha_3),(0,1,\alpha_3)\} \subseteq \mathcal{U}_{1,1},\ \alpha_i < 1\,, \forall i$, then one has:
a) for every $r > 5$, the gain g_r is not relevant;
b) if $\alpha_1 + \alpha_2 \leq 1$, then there exists a basic set $\mathcal{T}$, with $|\mathcal{T}| = 4$, and $\mathcal{A}_3$ is g-coherent.
c) if $\alpha_1 + \alpha_2 > 1$, then $\mathcal{A}_3$ is g-coherent iff

$$\alpha_3 \leq Max\{\frac{\alpha_1 + \alpha_2 - 2\alpha_1\alpha_2}{\alpha_1 + \alpha_2 - \alpha_1\alpha_2}\,,\ 1 - \alpha_1 + \alpha_1\alpha_3 - \alpha_1\alpha_2\alpha_3\,,\ 1 - \alpha_1\alpha_3\}\,.$$

Remark: We observe that, by suitably modifying the hypotheses in Theorems 4, 5, and 6, we obtain similar results on non relevant gains and basic sets, with further conditions characterizing the g-coherence of the assessment $\mathcal{A}_3$ on $\mathcal{F}_3$. As an example, by Theorem 4, still assuming $\mathcal{V}_0 = \mathcal{V}_1 = \mathcal{V}_2 = \emptyset$, under the hypotheses

$$\mathcal{U}_{0,1} = \{V_1, V_2\} = \{(1,1,0),(0,1,1)\},\ \{V_3, V_4\} = \{(1,0,\alpha_3),(\alpha_1,0,1)\} \subseteq \mathcal{U}_{1,1}\,,$$

we obtain a new result, which is similar to such theorem, and so on.

Theorem 7 If $\mathcal{V}_0 = \mathcal{V}_1 = \mathcal{V}_2 = \mathcal{U}_{0,1} = \emptyset,\ \mathcal{U}_{1,1} = \{V_1, V_2, V_3, V_4, V_5, V_6\} = \{(1,\alpha_2,0),(1,0,\alpha_3),(\alpha_1,1,0),(0,1,\alpha_3),(\alpha_1,0,1),(0,\alpha_2,1))\},\ \alpha_i < 1, \forall i$, then one has:
a) for every $r > 6$, the gain g_r is not relevant;
b) if $\alpha_1 + \alpha_2 \leq 1$, or $\alpha_1 + \alpha_3 \leq 1$, or $\alpha_2 + \alpha_3 \leq 1$, then there exists a basic set $\mathcal{T}$, with $|\mathcal{T}| = 4$, and $\mathcal{A}_3$ is g-coherent.
c) if $\alpha_1 + \alpha_2 > 1$, $\alpha_1 + \alpha_3 > 1$, $\alpha_2 + \alpha_3 > 1$, then $\mathcal{A}_3$ is not g-coherent.

Theorem 8 If $\mathcal{V}_0 = \mathcal{V}_1 = \mathcal{V}_2 = \mathcal{U}_{0,1} = \emptyset$, $\mathcal{U}_{1,1} = \{V_1, V_2, V_3, V_4\} = \{(1, \alpha_2, 0), (\alpha_1, 1, 0), (\alpha_1, 0, 1), (0, \alpha_2, 1))\}$, $\alpha_i < 1 \, \forall i$, then one has:
a) for every $r > 4$, the gain g_r is not relevant;
b) if $\alpha_1 + \alpha_3 \leq 1$, or $\alpha_2 + \alpha_3 \leq 1$, then there exists a basic set $\mathcal{T}$, with $|\mathcal{T}| = 2$, and $\mathcal{A}_3$ is g-coherent.
c) if $\alpha_1 + \alpha_3 > 1$, $\alpha_2 + \alpha_3 > 1$, then $\mathcal{A}_3$ is not g-coherent.

Theorem 9 If $\mathcal{V}_0 = \mathcal{V}_1 = \mathcal{V}_2 = \mathcal{U}_{0,1} = \emptyset$, $\mathcal{U}_{1,1} = \{V_1, V_2, V_3, V_4\} = \{(1, 0, \alpha_3), (0, 1, \alpha_3), (\alpha_1, 1, 0), (\alpha_1, 0, 1))\}$, $\alpha_i < 1 \, \forall i$, then one has:
a) for every $r > 4$, the gain g_r is not relevant;
b) if $\alpha_1 + \alpha_2 \leq 1$, or $\alpha_2 + \alpha_3 \leq 1$, then there exists a basic set $\mathcal{T}$, with $|\mathcal{T}| = 2$, and $\mathcal{A}_3$ is g-coherent.
c) if $\alpha_1 + \alpha_2 > 1$, $\alpha_2 + \alpha_3 > 1$, then $\mathcal{A}_3$ is not g-coherent.

Theorem 10 If $\mathcal{V}_0 = \mathcal{V}_1 = \mathcal{V}_2 = \mathcal{U}_{0,1} = \emptyset$, $\mathcal{U}_{1,1} = \{V_1, V_2, V_3, V_4\} = \{(1, 0, \alpha_3), (0, 1, \alpha_3), (1, \alpha_2, 0), (0, \alpha_2, 1))\}$, $\alpha_i < 1 \, \forall i$, then one has:
a) for every $r > 4$, the gain g_r is not relevant;
b) if $\alpha_1 + \alpha_2 \leq 1$, or $\alpha_1 + \alpha_3 \leq 1$, then there exists a basic set $\mathcal{T}$, with $|\mathcal{T}| = 2$, and $\mathcal{A}_3$ is g-coherent.
c) if $\alpha_1 + \alpha_2 > 1$, $\alpha_1 + \alpha_3 > 1$, then $\mathcal{A}_3$ is not g-coherent.

5 Some general results

In this section we give some theorems on g-coherence of a vector of lower probability bounds $\mathcal{A}_n$ defined on a family of n conditional events $\mathcal{F}_n$. Notice that detailed proofs of all theorems presented in this section are given in [6].
In the next theorem we generalize the condition 6 in Remark 4. In such theorem the set of logical relations L specifies that the conjunctions

$$\begin{array}{l} E_1H_1 \cdots E_nH_n \,, \quad E_1H_1 \cdots E_{n-1}H_{n-1}H_n^c \,, \quad \ldots \,, \quad H_1^c E_2H_2 \cdots E_nH_n \,, \\ E_1H_1 \cdots E_{n-2}H_{n-2}H_{n-1}^c H_n^c \,, \quad \ldots \,, \quad H_1^c H_2^c E_3H_3 \cdots E_nH_n \,, \quad \ldots \; \ldots \,, \\ E_1H_1H_2^c \cdots H_n^c \,, \quad \ldots \,, \quad H_1^c \cdots H_{n-1}^c E_nH_n \end{array}$$

are impossible. Then, under such hypotheses, the condition $\alpha_1 + \cdots + \alpha_n \leq n-1$ is necessary for the g-coherence of $\mathcal{A}_n$.

Theorem 11 If $\mathcal{V}_0 = \mathcal{V}_1 = \cdots = \mathcal{V}_{n-1} = \emptyset$ and $\alpha_1 + \cdots + \alpha_n > n-1$, then $\mathcal{A}_n$ is not g-coherent.

In the next theorem we generalize the condition 7 given in Remark 4.

Theorem 12 If $\mathcal{V}_0 = \cdots = \mathcal{V}_{n-1} = \emptyset$, $|\mathcal{U}_{0,1}| = n$, $0 < \alpha_i < 1 \; \forall i$, then one has:
a) there exists a basic set $\mathcal{T}$, with $|\mathcal{T}| = n$;
b) $\mathcal{A}_n$ is g-coherent iff $\alpha_1 + \cdots + \alpha_n \leq n-1$.

We denote by $\mathcal{Z}$ the set defined as

$$\mathcal{Z} = \{(h,k) : h+k = n-1 \ , \ h > 0\} \cup \{(h,k) : h+k < n-1\}\,.$$

Then, we have

Theorem 13 If $\mathcal{V}_0 = \cdots = \mathcal{V}_{n-1} = \emptyset$, $\mathcal{U}_{h,k} = \emptyset$ for each $(h,k) \in \mathcal{Z}$, and $\alpha_1 + \cdots + \alpha_n > 1$, then $\mathcal{A}_n$ is not g-coherent.

The next result generalizes the condition 8 in Remark 4.

Theorem 14 If $\mathcal{V}_0 = \cdots = \mathcal{V}_{n-1} = \emptyset$, $\mathcal{U}_{h,k} = \emptyset$, for each pair $(h,k) \in \mathcal{Z}$, $|\mathcal{U}_{0,n-1}| = n$, $0 < \alpha_i < 1 \; \forall\, i$, then one has:
a) if, for every $j \in J_n$, it is $\sum_{i \in J_n \setminus \{j\}} \alpha_i \leq 1$, then $\mathcal{T} = J_n$ is a basic set;
b) $\mathcal{A}_n$ is g-coherent iff $\alpha_1 + \cdots + \alpha_n \leq 1$.

6 Conclusions

Exploiting the coherence principle of de Finetti and the related notion of g-coherence, we illustrated a probabilistic approach to uncertain reasoning based on lower probability bounds. We examined the notions of non relevant gain and basic set which may be helpful, in g-coherence checking and propagation of conditional probability bounds, to reduce the sets of variables and/or constraints in the linear systems used in our algorithms. We observe that such notions and in particular the condition (6), in the form $g_r = \sum_{j \in \mathcal{T}_r} g_j$, have been used in ([3], Theorem 5.6) to characterize in term of random gains an efficient procedure proposed in [16] for families of conjunctive conditional events. To provide a better understanding of these notions, we examined several examples of families constituted by three conditional events. This case may have a specific importance, e.g., in default reasoning where many inference rules consist of two premises and one conclusion. We obtained some necessary and sufficient conditions of g-coherence and we also generalized some theoretical results. Further work should allow to extend the results of this paper to the case of families of n conditional events, with $n > 3$.

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