The Maximal Variance of Fuzzy Interval*

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Abstract

The paper gives the solution of calculating maximal variance of fuzzy interval in the scope of the theory of imprecise probabilities. As it appears, this problem is more difficult than analogous one connected with evaluation of lower and upper expectations of fuzzy interval. This paper gives some contribution to possibility theory in the framework of probability approach.

Keywords

possibility measure, upper and lower probabilities, maximal variance

1 Introduction

There is a well-known interpretation of fuzzy interval in the framework of the theory of imprecise probabilities [1, 2]. To get this, we associate with any fuzzy interval a possibility or necessity measure, and then consider that values of the pointed measures give us lower or upper assessments of probabilities. This interpretation was discussed in detail in [3], and there it is proposed to use upper and lower expectations for evaluating uncertainty of such intervals. These characteristics and other crude moments of order k can be easily calculated by Choquet integral. However, to calculate upper and lower central moments is more difficult as it is shown in investigations, presented below.

Throughout the paper we will use the following notations: 1) $E[\xi]$ is an ordinary expectation of the random variable ξ, i.e. $E[\xi] = \int\limits_{-\infty}^{+\infty} x dP(x)$, where P is a probability measure associated with the random variable ξ; 2) $\sigma^2[\xi]$ is an ordinary variance of the random variable ξ, i.e. $\sigma^2[\xi] = E\left[\xi^2\right] - (E[\xi])^2$.

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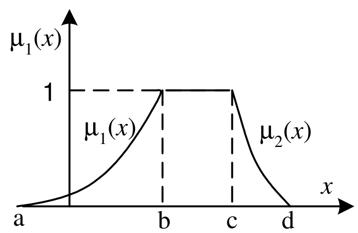

Figure 1: Membership function of a fuzzy interval

2 Basic definitions and problem statement

We will consider fuzzy intervals with a form (fig.1). The function μ is assumed to be continuous, and the functions μ_1 and μ_2 are differentiable on the intervals (a,b) and (c,d) correspondingly.

$$\mu(x) = \begin{cases} 0, & x \le a \ \ or \ \ x \ge d, \\ \mu_1(x), & a < x < b, \\ 1, & b \le x \le c, \\ \mu_2(x), & c < x < d. \end{cases} \tag{1}$$

In addition, μ_1 is increasing on (a,b), μ_2 is decreasing on (c,d).

In possibility theory, for each fuzzy interval, a possibility measure $\Pi(A) = \sup\limits_{x\in A}\mu(x)$ and a necessity measure $N(A) = \inf\limits_{x\notin A}[1-\mu(x)]$ are introduced, and can be considered as lower or upper estimation of probability of the event $A \in \Im$ (where $\Im$ is Borel algebra of real axis). Taking this into account, possibility measure Π and necessity measure N define a family of probability measures $\Xi = \{P\,|N(A) \le P(A) \le \Pi(A)\}$, and the problem arises, how to calculate digital characteristics of such family, in particular, the maximal variance $\overline{\sigma^2}(\mu) = \sup\limits_{P_i\in\Xi}\sigma^2[\xi_i]$. In the last expression, it is assumed that the probability measure P_i determines a random value ξ_i. For the fuzzy interval, the value $\overline{\sigma^2}(\mu)$ can serve as some characteristic of uncertainty.

3 The research of possibilistic inclusion

Theorem 1 [4, 5]. *Let P be a probability measure, Ξ a family of probability measures, generated by a fuzzy interval with a membership function μ. Then $P \in \Xi$ iff $P\{A(p)\} \le p$ for all $p \in [0,1]$, where $A(p) = \{x \in R\,|\mu(x) \le p\}$.*

Theorem 1 can be reformulated by using standard terms for random values as follows.

Theorem 1*. *Let we use the same notations as in theorem 1, and the random value* ξ *is described by the probability measure* P *on* $\Im$*. Consider also a random value* $\eta = \mu(\xi) \in [0,1]$*. Then* $P \in \Xi$ *iff* $F_\eta(y) \le y$*, where* $F_\eta(y) = P\{\eta \le y)$*.*

Remark. The function F_η is a distribution function of η, whenever η is continuous.

4 The solution of the optimization problem

Theorem 2. *Let* ξ *be a random value, described by a probability measure* $P \in \Xi$*, in addition,* $\sigma^2[\xi] = \overline{\sigma^2}(\mu)$*. Then we have* $P\{(b,c)\} = 0$ *for fuzzy interval (1).*

Proof. Suppose that the coordinate system has been chosen in a way that $E[\xi] = 0$. Assume also that $b < 0$, and the condition of the theorem is not fulfilled, i.e. $P(b,0] > 0$. The theorem is valid if one can find such a measure P^* that $P^* \in \Xi$ and $\sigma^2[\xi^*] > \sigma^2[\xi]$. We will search the probability measure P^* in a form:

$$P^*(A) = \begin{cases} P(A)\varepsilon, & A \subseteq (b,0], \\ P\{b\} + P(b,0](1-\varepsilon), & A = \{b\}, \\ P(A), & A \cap [b,0] = \emptyset. \end{cases}$$

It is obvious that P^* extends on $\Im$ uniquely and $P^* \in \Xi$. Calculate derivative of

$$\sigma^2[\xi^*] = \int_{-\infty}^{+\infty} x^2 dP^*(x) - \left[\int_{-\infty}^{+\infty} x dP^*(x)\right]^2$$

w.r.t. ε at the point $\varepsilon = 1$. Since $\int_{-\infty}^{+\infty} x dP^*(x) = 0$ at the point $\varepsilon = 1$,

$$\frac{d}{d\varepsilon}\left(\sigma^2[\xi^*]\right)_{\varepsilon=1} = \frac{d}{d\varepsilon}\left[\int_{-\infty}^{+\infty} x^2 dP^*(x)\right]_{\varepsilon=1}.$$

Describe the last expression in detail.

$$\int_{-\infty}^{+\infty} x^2 dP^*(x) = \int_{R\setminus[b,0]} x^2 P(x) + b^2\left(P\{b\} + P(b,0](1-\varepsilon)\right) + \varepsilon \int_{(b,0]} x^2 dP(x).$$

Therefore,

$$\frac{d}{d\varepsilon}\left(\sigma^2[\xi^*]\right)_{\varepsilon=1} = -b^2 P(b,0] + \int_{(b,0]} x^2 dP(x) < 0.$$

It means that there exists $\varepsilon < 1$ that $\sigma^2[\xi^*] > \sigma^2[\xi]$. For the complete proof of the theorem, we must consider also a case, where $c > 0$ and $P[0,c) > 0$.

Corollary. *Let $P \in \Xi$, $\sigma^2[\xi] = \overline{\sigma^2}(\mu)$ as in theorem 2, in addition, $E[\xi] = 0$. Then*

1) $P[0,c) = 0$ if $c > 0$;

2)$P(b,0) = 0$ if $b < 0$.

Theorem 3. *Let $P \in \Xi$, $\sigma^2[\xi] = \overline{\sigma^2}(\mu)$ as in theorem 2. Then the random value η has a distribution function $F_\eta(y) = y$.*

Proof. We will assume that the coordinate system has been chosen in a way that $E[\xi] = 0$. Suppose the contrary assumption, that for ξ from the theorem, $F_\eta(y) \neq y$. The theorem will be proved, if under this condition, there exists a random value ξ^* associated with a probability measure $P^* \in \Xi$ for which $\sigma^2[\xi^*] > \sigma^2[\xi]$. The random value ξ^* will be searched for a certain $\alpha \in [0,1]$, using the expression:

$$\xi^* = \begin{cases} \mu_1^{-1}\left[\alpha F_\eta\left(\mu(\xi)\right) + (1-\alpha)\mu(\xi)\right], & \xi \in [a,b], \\ \mu_2^{-1}\left[\alpha F_\eta\left(\mu(\xi)\right) + (1-\alpha)\mu(\xi)\right], & \xi \in [c,d]. \end{cases}$$

Hence, we need to find $\alpha \in [0,1]$ such that $\sigma^2[\xi^*] > \sigma^2[\xi]$. But at first, check that ξ^* generates the probability measure $P^* \in \Xi$. To do this, we need to confirm that the inequality

$$F_{\eta^*}(y) = P\left\{\mu(\xi^*) \leq y\right\} \leq y$$

is valid. Actually,

$$\eta^* = \mu(\xi^*) = \alpha F_\eta\left(\mu(\xi)\right) + (1-\alpha)\mu(\xi),$$

$$F_{\eta^*}(y) = P\left\{\alpha F_\eta\left(\mu(\xi)\right) + (1-\alpha)\mu(\xi) \leq y\right\}.$$

Since $F_\eta(y) \leq y$, then $\left\{\alpha F_\eta\left(\mu(\xi)\right) + (1-\alpha)\mu(\xi) \leq y\right\} \subseteq \left\{F_\eta\left(\mu(\xi)\right) \leq y\right\}$. Therefore,

$$F_\eta(y) \leq P\left\{F_\eta\left(\mu(\xi)\right) \leq y\right\} = P\left\{\mu(\xi) \leq F_\eta^{-1}(y)\right\} = F_\eta\left(F_\eta^{-1}(y)\right) = y.$$

Thus, it has been shown that $P^* \in \Xi$. Further we will prove that $\sigma^2[\xi^*] > \sigma^2[\xi]$ for a certain $\alpha \in [0,1]$. To do this, calculate derivative of

$$\frac{d}{d\alpha}\sigma^2\left[\xi^*\right] = \frac{d}{d\alpha}\left(E\left[(\xi^*)^2\right] - \left(E\left[\xi^*\right]\right)^2\right)_{\alpha=0}$$

at the point $\alpha = 0$. Since $E\left[\xi^*\right]_{\alpha=0} = E\left[\xi\right] = 0$,

$$\left.\frac{d}{d\alpha}\sigma^2\left[\xi^*\right]\right|_{\alpha=0} = \left.\frac{d}{d\alpha}E\left[(\xi^*)^2\right]\right|_{\alpha=0} =$$

$$\frac{d}{d\alpha}\left(\int_a^b \left[\mu_1^{-1}\left[\alpha F_\eta\left(\mu(x)\right) + (1-\alpha)\mu(x)\right]\right]^2 dP(x)\right)_{\alpha=0} +$$

$$+\frac{d}{d\alpha}\left(\int_{c}^{d}\left[\mu_2^{-1}\left[\alpha F_\eta\left(\mu(x)\right)+(1-\alpha)\mu(x)\right]\right]^2 dP(x)\right)_{\alpha=0}.$$

Taking derivative w.r.t. α at the point $\alpha=0$, we get

$$\frac{d}{d\alpha}\sigma^2\left[\xi^*\right]\Big|_{\alpha=0}=\int_a^b 2x\,\frac{d}{dy}\mu_1^{-1}(y)\Big|_{y=\mu(x)}\left(F_\eta\left(\mu(x)\right)-\mu(x)\right)dP(x)+$$

$$+\int_c^d 2x\,\frac{d}{dy}\mu_2^{-1}(y)\Big|_{y=\mu(x)}\left(F_\eta\left(\mu(x)\right)-\mu(x)\right)dP(x).$$

Analyze signs of factors stating in the integrands.

1) $F_\eta\left(\mu(x)\right)-\mu(x)\le 0$, in addition, since according to our supposition $F_\eta(y)\ne y$, $y\in[0,1]$, there exists a non-empty set of points, in which $F_\eta\left(\mu(x)\right)-\mu(x)<0$. Since F_η is continuous, increasing function, $P\left\{F_\eta\left(\mu(\xi)\right)-\mu(\xi)<0\right\}>0$.

2) The function μ_1 is increasing on $[a,b]$, therefore, $\frac{d}{dy}\mu_1^{-1}(y)\Big|_{y=\mu(x)}>0$ if $x\in(a,b)$.

3) According to the corollary of theorem 2, $P[0,c)=0$. It enables to exchange the area of integration to $(a,\min\{b,0\})$. Notice that $(2x)<0$ if x is in this interval.

4) The function μ_2 is decreasing on $[c,d]$, thus, $\frac{d}{dy}\mu_2^{-1}(y)\Big|_{y=\mu(x)}<0$, whenever $x\in(c,d)$.

5) According to the corollary of theorem 2, $P[d,0)=0$. It enables to exchange the area of integration to $(\max\{c,0\},d)$ in the second integral. Note that the factor $(2x)>0$ in this interval.

Analyzing signs of integrals, one can confirm that value of each of them is non-negative; in addition, one of them is strictly positive. Hence, $\frac{d}{d\alpha}\sigma^2\left[\xi^*\right]\big|_{\alpha=0}>0$. It means that one can find $\alpha>0$ that $\sigma^2[\xi^*]>\sigma^2[\xi]$, i.e. the supposition has been made is wrong, and it implies $F_\eta(y)=y$.

The proved theorems enable to make some simplifying of our optimization problem. To do this, introduce into consideration the functions

$$F_{\eta_1}(y)=P\left\{\mu_1(\xi)\le y\,|\xi\in[a,b]\right\},F_{\eta_2}(y)=P\left\{\mu_2(\xi)\le y\,|\xi\in[c,d]\right\}.$$

It is clear that $\eta_1=\mu(\xi_1)$, $\eta_2=\mu(\xi_2)$, and also the random value ξ_1 is associated with the probability measure $P\{*|\xi\in[a,b]\}$, ξ_2 with the probability measure $P\{*|\xi\in[c,d]\}$. Let $P\in\Xi$ and $\sigma^2[\xi]=\overline{\sigma^2}(\mu)$, then $P\{R\backslash[a,b]\cup[c,d]\}=0$, and, using formula of composite probability, one can write:

$$F_\eta(y)=F_{\eta_1}(y)P\left\{\xi\in[a,b]\right\}+F_{\eta_2}(y)P\left\{\xi\in[c,d]\right\}.$$

By theorem 3, $F_\eta(y) = y$. Assume that functions $F_{\eta_1}(y)$ and $F_{\eta_2}(y)$ are differentiable, then the calculation of probability is transformed to Riemannian integral:

$$P\{\xi \in A\} = P\{\xi \in [a,b]\} \int\limits_{\mu\{A\cap[a,b]\}} dF_{\eta_1}(y) + P\{\xi \in [c,d]\} \int\limits_{\mu\{A\cap[c,d]\}} dF_{\eta_2}(y). \tag{2}$$

By analogy, using the expression $\xi = \begin{cases} \mu_1^{-1}(\eta_1), \xi \in [a,b], \\ \mu_2^{-1}(\eta_2), \xi \in [c,d], \end{cases}$ one can write the formula for calculating moments:

$$E\left[\xi^k\right] = P\{\xi \in [a,b]\} \int_0^1 \left[\mu_1^{-1}(y)\right]^k dF_{\eta_1}(y) + P\{\xi \in [c,d]\} \int_0^1 \left[\mu_2^{-1}(y)\right]^k dF_{\eta_2}(y).$$

Introduce the following notations:

$$h_1(y) = P\{\xi \in [a,b]\} F'_{\eta_1}(y), \qquad h_2(y) = P\{\xi \in [c,d]\} F'_{\eta_2}(y).$$

Then

$$E\left[\xi^k\right] = \int_0^1 \left[\mu_1^{-1}(y)\right]^k h_1(y)dy + \int_0^1 \left[\mu_2^{-1}(y)\right]^k h_2(y)dy. \tag{3}$$

It is clear, that functions h_1, h_2 have to be non-negative in $[0,1]$, in addition, $h_1(y) + h_2(y) = 1$ by theorem 3.

Theorem 4. *Let ξ be associated with a probability measure P, $P \in \Xi$, $\sigma^2[\xi] = \overline{\sigma^2}(\mu)$, $E[\xi] = 0$, and $E\left[\xi^k\right]$ is calculated by formula (3). In addition, functions h_1, h_2 are piecewise continuous. Then in the range of h_i continuity the following formula is valid:*

$$h_1(y) = \begin{cases} 1, \left|\mu_1^{-1}(y)\right| > \left|\mu_2^{-1}(y)\right|, \\ 0, \left|\mu_1^{-1}(y)\right| < \left|\mu_2^{-1}(y)\right|, \end{cases} \qquad h_2(y) = 1 - h_1(y). \tag{4}$$

Proof. Assume, on the contrary, that the condition of the theorem takes place, but formula (4) is not valid at least for one point of $h_i(y)$ continuity. The theorem is valid if, for this case, we can find a random value ξ^*, associated with a probability measure $P^* \in \Xi$ such that $\sigma^2[\xi^*] > \sigma^2[\xi]$. To do this, introduce into consideration the following functions:

$$g_1(y) = \begin{cases} 1, & \left|\mu_1^{-1}(y)\right| > \left|\mu_2^{-1}(y)\right|, \\ 0, & \left|\mu_1^{-1}(y)\right| < \left|\mu_2^{-1}(y)\right|, \\ h_1(y), & \left|\mu_1^{-1}(y)\right| = \left|\mu_2^{-1}(y)\right|, \end{cases} \qquad g_2(y) = 1 - g_1(y),$$

and also

$$h_1^*(y) = g_1(y)\alpha + h_1(y)(1-\alpha), \quad h_2^*(y) = g_2(y)\alpha + h_2(y)(1-\alpha), \;\; y \in [0,1].$$

It is assumed that functions h_1^*, h_2^* generate the probability distribution of ξ^* by the formula:

$$P\{\xi^* \in A\} = \int\limits_{\mu\{A\cap[a,b]\}} h_1^*(y)dy + \int\limits_{\mu\{A\cap[c,d]\}} h_2^*(y)dy.$$

It is clear that the last formula is an analog of formula (2), and the random value ξ^* generates the probability measure $P^* \in \Xi$ for all values $\alpha \in [0,1]$. Calculate derivative of $\frac{d}{d\alpha}\sigma^2[\xi^*] = \frac{d}{d\alpha}\left(E\left[(\xi^*)^2\right] - (E[\xi^*])^2\right)_{\alpha=0}$ at the point $\alpha = 0$. Since $E[\xi^*] = 0$ for $\alpha = 0$, we get $\frac{d}{d\alpha}\sigma^2[\xi^*]\big|_{\alpha=0} = \frac{d}{d\alpha}E\left[(\xi^*)^2\right]\Big|_{\alpha=0}$. Then

$$\frac{d}{d\alpha}E\left[(\xi^*)^2\right] = \frac{d}{d\alpha}\int\limits_0^1 \left[\mu_1^{-1}(y)\right]^2 [g_1(y)\alpha + h_1(y)(1-\alpha)]\,dy +$$

$$+\frac{d}{d\alpha}\int\limits_0^1 \left[\mu_2^{-1}(y)\right]^2 [g_2(y)\alpha + h_2(y)(1-\alpha)]\,dy =$$

$$= \int\limits_0^1 \left[\mu_1^{-1}(y)\right]^2 [g_1(y) - h_1(y)]\,dy + \int\limits_0^1 \left[\mu_2^{-1}(y)\right]^2 [g_2(y) - h_2(y)]\,dy.$$

Since $g_1(y) - h_1(y) = h_2(y) - g_2(y)$, we get at last

$$\frac{d}{d\alpha}\sigma^2[\xi^*]\Big|_{\alpha=0} = \int\limits_0^1 \left(\left[\mu_1^{-1}(y)\right]^2 - \left[\mu_2^{-1}(y)\right]^2\right)[g_1(y) - h_1(y)]\,dy.$$

Analyze signs of integrands factors.

1) Let $g_1(y) > h_1(y)$, then $g_1(y) = 1$, i.e. $\mu_1^{-1}(y) > \mu_2^{-1}(y)$ by formula (4).

2) Let $g_1(y) < h_1(y)$, then $g_1(y) = 0$. i.e. $\mu_1^{-1}(y) < \mu_2^{-1}(y)$ by formula (4).

From this, one can make a conclusion that the integrand on $[0,1]$ is non-negative. In addition, by our assumption, there is a point in the range of $h_1(y)$ continuity such that $g_1(y) \neq h_1(y)$. It implies $\frac{d}{d\alpha}\sigma^2[\xi^*]\big|_{\alpha=0} > 0$. It means, there is a point $\alpha > 0$ such that $\sigma^2[\xi^*] > \sigma^2[\xi]$, i.e. the assumption made is wrong. It proves the theorem in the whole.

Theorem 5. *Let a set $\{\xi_i\}$ of random values with maximal variance $\sigma^2[\xi_i] = \overline{\sigma^2}(\mu)$, $E[\xi_i] = 0$, be in a fuzzy interval F with the membership function μ. Then there is a random value $\xi^* = \{\xi_i\}$ such that*

$$h_1^*(y) = \begin{cases} 1, \left|\mu_1^{-1}(y)\right| > \left|\mu_2^{-1}(y)\right|, \\ 0, \left|\mu_1^{-1}(y)\right| < \left|\mu_2^{-1}(y)\right|, \\ \alpha, \left|\mu_1^{-1}(y)\right| = \left|\mu_2^{-1}(y)\right|, \end{cases} \qquad h_2^*(y) = 1 - h_1^*(y), \quad \alpha \in [0,1].$$

Proof. By theorem 4, the set $\{\xi_i\}$ includes a random value ξ such that

$$h_1(y) = \begin{cases} 1, \left|\mu_1^{-1}(y)\right| > \left|\mu_2^{-1}(y)\right|, \\ 0, \left|\mu_1^{-1}(y)\right| < \left|\mu_2^{-1}(y)\right|, \end{cases} \qquad h_2(y) = 1 - h_1(y).$$

Denote $A = \left\{y \in [0,1] \middle| \left|\mu_1^{-1}(y)\right| = \left|\mu_2^{-1}(y)\right|\right\}$. For the random value ξ^*, choose parameter $\alpha \in [0,1]$ as follows. Under the condition, $E[\xi] = E[\xi^*] = 0$, in addition, $h_i(y) = h_i^*(y)$ if $y \in \bar{A}$. Therefore,

$$E[\xi^*] - E[\xi] = \int_A \mu_1^{-1}(y)h_1(y)dy + \int_A \mu_2^{-1}(y)h_2(y)dy -$$

$$-\left(\int_A \mu_1^{-1}(y)h_1^*(y)dy + \int_A \mu_2^{-1}(y)h_2^*(y)dy\right).$$

For $y \in A$, $\mu_1^{-1}(y) = -\mu_2^{-1}(y)$, thus,

$$\int_A \mu_1^{-1}(y)h_1(y)dy + \int_A \mu_2^{-1}(y)h_2(y)dy =$$

$$= \int_A \mu_1^{-1}(y)\left(h_1(y) - h_2(y)\right)dy = \beta \int_A \mu_1^{-1}(y)dy,$$

where $\beta \in [0,1]$. The last equality is obtained with the help of mean-value theorem. By analogy,

$$\int_A \mu_1^{-1}(y)h_1^*(y)dy + \int_A \mu_2^{-1}(y)h_2^*(y)dy =$$

$$= \int_A \mu_1^{-1}(y)\left(h_1^*(y) - h_2^*(y)\right)dy = (2\alpha - 1)\int_A \mu_1^{-1}(y)dy.$$

Thus, $E[\xi] = E[\xi^*] = 0$ if $\beta = 2\alpha - 1$. Let us show that $\sigma^2[\xi^*] = \sigma^2[\xi]$ in this case. Actually,

$$\sigma^2[\xi] - \sigma^2[\xi^*] = E\left[\xi^2\right] - E\left[(\xi^*)^2\right] =$$

$$-\left(\int_A \left[\mu_1^{-1}(y)\right]^2 h_1^*(y)dy + \int_A \left[\mu_2^{-1}(y)\right]^2 h_2^*(y)dy\right) =$$

$$= \int_A \left[\mu_1^{-1}(y)\right]^2 dy - \int_A \left[\mu_2^{-1}(y)\right]^2 dy = 0.$$

The theorem is proved.

5 The practical calculation of maximal variance

Theorem 6. *Let the function $\mu_1^{-1} + \mu_2^{-1}$ be increasing. Then functions h_i for calculating the maximal variance have a form:*

$$h_1(y) = \begin{cases} 1, y < \alpha, \\ 0, y > \alpha, \end{cases} \qquad h_2(y) = 1 - h_1(y), \quad y, \alpha \in [0,1]. \tag{5}$$

Proof. Let ξ be associated with a probability measure $P \in \Xi$ and $\sigma^2[\xi_i] = \overline{\sigma^2}(\mu)$. Suppose that $E[\xi] = m$, then by theorem 4,

$$h_1(y) = \begin{cases} 1, \left|\mu_1^{-1}(y) - m\right| > \left|\mu_2^{-1}(y) - m\right|, \\ 0, \left|\mu_1^{-1}(y) - m\right| < \left|\mu_2^{-1}(y) - m\right|. \end{cases}$$

Thus, we need to solve the inequality, $\left|\mu_1^{-1}(y) - m\right| > \left|\mu_2^{-1}(y) - m\right|$. One can consider that $\mu_1^{-1}(y) - m < 0$ and $\mu_2^{-1}(y) - m > 0$ (see corollary of theorem 2). Therefore, the last inequality is transformed to a form:

$$\mu_1^{-1}(y) + \mu_2^{-1}(y) < 2m.$$

Let the number $2m$ belong to the range of values of the function $\mu_1^{-1} + \mu_2^{-1}$, g be an inverse function to this median, then, since g is increasing function, we get that $y < g(2m) = \alpha$. The cases, where $2m$ does not belong to the range of median values, are also described by formula (5).

Corollaries of theorem 6. *Let we use notations of theorem 6. Then*

1) $h_1(y) = \begin{cases} 1, \mu_1^{-1}(y) + \mu_2^{-1}(y) < 0, \\ 0, \mu_1^{-1}(y) + \mu_2^{-1}(y) > 0, \end{cases}$ $h_2(y) = 1 - h_1(y)$, *if* $E[\xi] = 0$.

2) Let the function $\mu_1^{-1} + \mu_2^{-1}$ be increasing on $[0,1]$, then there is a certain $\alpha \in [0,1]$ such that $h_1(y) = \begin{cases} 1, y > \alpha, \\ 0, y < \alpha, \end{cases}$ $h_2(y) = 1 - h_1(y)$, $y, \alpha \in [0,1]$.

Theorem 7. *Let ξ belong to a fuzzy interval F with a membership function μ and $\sigma^2[\xi] = \overline{\sigma^2}(\mu)$. Then $E[\xi] \in \left\{\mu_1^{-1}(y) + \mu_2^{-1}(y) \,|\, y \in [0,1]\right\}$.*

Proof. Assume that the condition of the theorem is not satisfied. Then, using corollary 1 of theorem 6, we get that either $h_1(y) \equiv 1$ or $h_2(y) \equiv 1$. For the sake of determinacy, let $E[\xi] = 0$. 1) Let $h_1(y) \equiv 1$, then $E[\xi] < b$. It means that $P[0, c) > 0$, but this contradicts to the corollary of theorem 2. 2) Let $h_2(y) \equiv 1$, then $E[\xi] > c$. It means that $P(b, 0] > 0$, but this contradicts to the corollary of theorem 2. The contradictions found prove the truth of the theorem.

Corollary. *Let a fuzzy interval F be symmetric, i.e. $\mu_1^{-1}(y) + \mu_2^{-1}(y) = const$, and, for the sake of determinacy, $const = 0$. Then $\overline{\sigma^2}(\mu) = \int_0^1 \left[\mu_1^{-1}(y)\right]^2 dy$.*

Proof. According to theorem 7, for ξ with maximal variance, the value $E[\xi]$

belongs to the range of $\mu_1^{-1}+\mu_2^{-1}$ values, i.e. $E[\xi]=const=0$. Therefore,

$$\sigma^2[\xi]=\int_0^1\left[\mu_1^{-1}(y)\right]^2 h_1(y)dy+\int_0^1\left[\mu_2^{-1}(y)\right]^2 h_2(y)dy.$$

Since $\left[\mu_1^{-1}(y)\right]^2=\left[\mu_2^{-1}(y)\right]^2$ and $h_1(y)+h_2(y)=1$, we get $\sigma^2[\xi]=\int_0^1\left[\mu_1^{-1}(y)\right]^2dy$. The corollary is proved.

Theorem 8. *Let functions $h_i(y)$ in the formula (3) for calculating maximal variance have a form:*

$$h_1(y)=\begin{cases}1, y<\alpha,\\ 0, y>\alpha,\end{cases}\qquad h_2(y)=1-h_1(y),\quad y,\alpha\in[0,1]. \tag{6}$$

Then α can be found from the equality:

$$\frac{\mu_1^{-1}(\alpha)+\mu_2^{-1}(\alpha)}{2}+\int_0^\alpha\left[\mu_2^{-1}(y)-\mu_1^{-1}(y)\right]dy-\frac{1}{2}\int_0^1\left[\mu_2^{-1}(y)-\mu_1^{-1}(y)\right]dy=0, \tag{7}$$

if the coordinate system is chosen such that $\int_0^1\left[\mu_2^{-1}(y)+\mu_1^{-1}(y)\right]dy=0$. There is a unique solution if the function $\mu_1^{-1}+\mu_2^{-1}$ is increasing.

Proof. Let the functions h_i have a form (6). Then

$$\overline{\sigma^2}[\mu]=E\left[\xi^2\right]-E^2[\xi]=\int_0^\alpha\left[\mu_1^{-1}(y)\right]^2 h_1(y)dy+$$

$$+\int_\alpha^1\left[\mu_2^{-1}(y)\right]^2 h_2(y)dy-\left(\int_0^\alpha\mu_1^{-1}(y)dy+\int_\alpha^1\mu_2^{-1}(y)dy\right)^2.$$

Taking derivative w.r.t. α and using the necessity condition for extremum, we get the equality:

$$\left[\mu_1^{-1}(\alpha)\right]^2-\left[\mu_2^{-1}(\alpha)\right]^2-2\left[\mu_1^{-1}(\alpha)-\mu_2^{-1}(\alpha)\right]\left[\int_0^\alpha\mu_1^{-1}(y)dy+\int_\alpha^1\mu_2^{-1}(y)dy\right]=0.$$

Since $\alpha<1$ and $\mu_1^{-1}(\alpha)-\mu_2^{-1}(\alpha)<0$, then we can reduce this factor. As result,

$$\mu_1^{-1}(\alpha)+\mu_2^{-1}(\alpha)-2\left[\int_0^\alpha\mu_1^{-1}(y)dy+\int_\alpha^1\mu_2^{-1}(y)dy\right]=0. \tag{8}$$

Transform the expression:

$$2\left[\int_0^{\alpha}\mu_1^{-1}(y)dy+\int_{\alpha}^{1}\mu_2^{-1}(y)dy\right]=\int_0^{\alpha}\mu_1^{-1}(y)dy-\int_0^{\alpha}\mu_2^{-1}(y)dy+\int_0^{1}\mu_2^{-1}(y)dy$$

$$+\int_0^{1}\mu_1^{-1}(y)dy-\int_{\alpha}^{1}\mu_1^{-1}(y)dy++\int_{\alpha}^{1}\mu_2^{-1}(y)dy=\left(\int_{\alpha}^{1}\left[\mu_2^{-1}(y)-\mu_1^{-1}(y)\right]dy-\right.$$

$$\left.-\int_0^{\alpha}\left[\mu_2^{-1}(y)-\mu_1^{-1}(y)\right]dy\right)+\int_0^{1}\left[\mu_2^{-1}(y)+\mu_1^{-1}(y)\right]dy.$$

By the supposition, $\int_0^1\left[\mu_2^{-1}(y)+\mu_1^{-1}(y)\right]dy=0$, in addition, the first item in the last expression can be transformed to a form:

$$\int_0^{1}\left[\mu_2^{-1}(y)-\mu_1^{-1}(y)\right]dy-2\int_0^{\alpha}\left[\mu_2^{-1}(y)-\mu_1^{-1}(y)\right]dy.$$

Taking this into account, the equality (8) is written as follows:

$$\mu_1^{-1}(\alpha)+\mu_2^{-1}(\alpha)+2\int_0^{\alpha}\left[\mu_2^{-1}(y)-\mu_1^{-1}(y)\right]dy--\int_0^{1}\left[\mu_2^{-1}(y)-\mu_1^{-1}(y)\right]dy=0, \tag{9}$$

i.e. we really prove the truth of equation (7).

Denote the left part of equation (9) by $f(\alpha)$. Let the function $\mu_1^{-1}+\mu_2^{-1}$ be increasing, then, since by supposition $\int_0^1\left[\mu_2^{-1}(y)+\mu_1^{-1}(y)\right]dy=0$, it is obvious that $\mu_1^{-1}(0)+\mu_2^{-1}(0)\leq 0$ and $\mu_1^{-1}(1)+\mu_2^{-1}(1)\geq 0$. Taking this into our account, analyze signs of $f(\alpha)$ at the ends of $[0,1]$:

$$f(0)=\mu_1^{-1}(0)+\mu_2^{-1}(0)-S,$$

$$f(1)=\mu_1^{-1}(1)+\mu_2^{-1}(1)+S,$$

where S is an area of the fuzzy interval. Therefore, $f(0)<0$ and $f(1)>0$, i.e. the equation has at least one root. Analyze the sign of

$$f'(\alpha)=\frac{d}{d\alpha}\left[\mu_1^{-1}(\alpha)+\mu_2^{-1}(\alpha)\right]+2\left[\mu_2^{-1}(\alpha)-\mu_1^{-1}(\alpha)\right].$$

It is obvious, that $f'(\alpha)>0$ for $\alpha\in[0,1]$. Thus, the equality $f(\alpha)=0$ has only one root, and this root is a point of maximum (you should remind, that for obtaining equality (8), we reduce the expression by the negative factor $\left(\mu_1^{-1}(\alpha)-\mu_2^{-1}(\alpha)\right)$). Thus, the theorem is proved in the whole.

Remarks.

1) Theorem 8 is easily generalized for the case, where

$$h_1(y)=\begin{cases}1, y>\alpha,\\ 0, y<\alpha,\end{cases}\quad h_2(y)=1-h_1(y),\ \ y,\alpha\in[0,1],$$

and the function $\mu_1^{-1}+\mu_2^{-1}$ is decreasing. In this case α can be found from the equation:

$$\frac{\mu_1^{-1}(\alpha)+\mu_2^{-1}(\alpha)}{2}-\int_0^{\alpha}\left[\mu_2^{-1}(y)-\mu_1^{-1}(y)\right]dy+\frac{1}{2}\int_0^{1}\left[\mu_2^{-1}(y)-\mu_1^{-1}(y)\right]dy=0.$$

We also suppose that $\int_0^1\left[\mu_2^{-1}(y)+\mu_1^{-1}(y)\right]dy=0$.

2) The equation (7) has a geometrical interpretation (fig. 2).

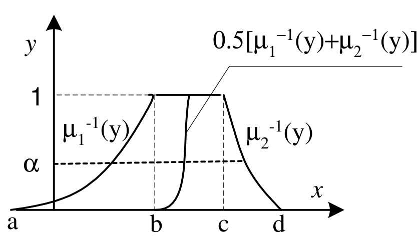

Figure 2: Fuzzy interval: inverse functions

a) $0.5(\mu_1^{-1}+\mu_2^{-1})$ is the median of the fuzzy interval;

b) $\int_0^1\left[\mu_2^{-1}(y)-\mu_1^{-1}(y)\right]dy$ is the area of the fuzzy interval;

c) $\int_0^{\alpha}\left[\mu_2^{-1}(y)-\mu_1^{-1}(y)\right]dy$ is the area of the part of the fuzzy interval that is below of α level ;

d) $\mu_2^{-1}(\alpha)-\mu_1^{-1}(\alpha)$ is the length of the level line $y=\alpha$ for the fuzzy interval.

3) Introduce into consideration functions

$$m(y)=\frac{\mu_1^{-1}(y)+\mu_2^{-1}(y)}{2},\ w(y)=\frac{\mu_2^{-1}(y)-\mu_1^{-1}(y)}{2},\ F(\alpha)=\int_0^{\alpha}\left[w(y)+\frac{m'(y)}{2}\right]dy.$$

Then equation (7) can be transformed to a form:

$$2F(\alpha)-F(1)=0. \tag{7*}$$

Example. Consider, how to calculate the maximal variance for the fuzzy interval having a form of trapezium (fig. 3). In this case, the functions m, w are linear.

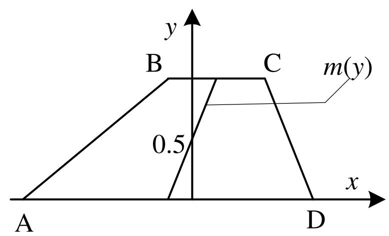

Figure 3: Fuzzy interval with a form of trapezium

We assume that m is increasing and $\int_0^1 m(y)dy = 0$. In this case, one can easily show that $m(y) = k(y-0.5)$, where $k > 0$. The function w is expressed through lengths of the trapezium sides $l_1 = |BC|$ and $l_2 = |AD|$. Since $w(0) = 0.5l_2$ and $w(1) = 0.5l_1$, then $w(y) = 0.5[l_2 - (l_2 - l_1)y]$. The parameter α can be found from equation (7*). Then $F(\alpha) = 0.5(k + l_2)\alpha - 0.25(l_2 - l_1)\alpha^2$, and we need to solve the equation:

$$(l_2 - l_1)\alpha^2 - 2(k + l_2)\alpha + \frac{l_1 + l_2 + 2k}{2} = 0.$$

Solving it, we get

$$\alpha = \frac{(k + l_2) - \sqrt{0.5\left[(k + l_2)^2 + (k + l_1)^2\right]}}{l_2 - l_1},$$

in addition, $\alpha \in [0,1]$. The precise value of $\overline{\sigma^2}(\mu)$ can be calculated by formula (3). Namely, according to the form of h_1, h_2 we can write

$$\overline{\sigma^2}(\mu) = \int_0^\alpha \left[\mu_1^{-1}(y)\right]^2 dy + \int_\alpha^1 \left[\mu_2^{-1}(y)\right]^2 dy - \left(\int_0^\alpha \mu_1^{-1}(y)dy + \int_\alpha^1 \mu_2^{-1}(y)dy\right)^2,$$

where

$$\mu_1^{-1}(y) = [k + 0.5(l_2 - l_1)](y - 0.5) - 0.25(l_2 + l_1),$$

$$\mu_2^{-1}(y) = [k - 0.5(l_2 - l_1)](y - 0.5) + 0.25(l_2 + l_1).$$

Let $l_1 = 1,\ l_2 = 3,\ k = 0.5$, then $\alpha = 0.404, \overline{\sigma^2}(\mu) = 1.342$. Fig. 4 shows this fuzzy interval, and probability distribution function F of the extreme random value ξ, being in the fuzzy interval, for which $\overline{\sigma^2}(\mu) = \sigma^2[\xi]$.

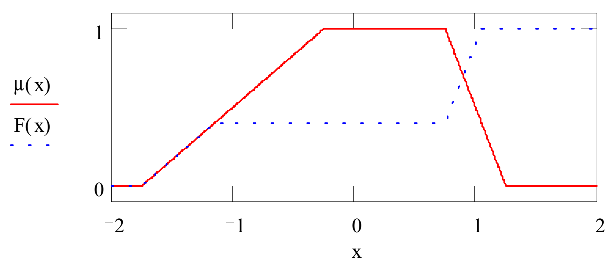

Figure 4: Numerical example

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