Expected Utility with Multiple Priors*

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Abstract

Let $\succsim$ be a preference relation on a convex set F. Necessary and sufficient conditions are given that guarantee the existence of a set $\{u_l\}$ of affine utility functions on F such that $\succsim$ is represented by $U(f) = u_l(f)$ if $f \in F_l$; where each F_l is a convex subset of F. The interpretation is simple: facing a "non-homogeneous" set F of alternatives, a decision maker splits it into "homogeneous" subsets F_l, and acts as a standard expected utility maximizer on each of them.

In particular, when F is a set of simple acts, each u_l corresponds to a subjective expected utility with respect to a finitely additive probability P_l; while when F is a set of continuous acts, each probability P_l is countably additive.

Keywords

preference representation, subjective probability, nonexpected utility, integral representation, multiple priors, countable additivity

1 Introduction

Given a preference relation $\succsim$ on a convex set F, we provide necessary and sufficient conditions that guarantee the existence of a set $\{u_l\}$ of affine utility functions on F such that $\succsim$ is represented by

$$U(f) = u_l(f) \quad \text{if } f \in F_l,$$

*We gratefully acknowledge the financial support of CNR and MIUR. First version: November 1999.

where each F_l is a convex subset of F. This representation has a simple interpretation: facing a "non-homogeneous" set of alternatives F, a decision maker splits it into "homogeneous" subsets F_l and, on each of them, she behaves as a standard expected utility maximizer. For example, the F_l can be commodities traded in a local market l and F be the global market, or the F_l can be sets of lotteries on which the decision maker feels she has the same information.

The idea underlying these results is close to the one of Castagnoli and Maccheroni (2000), but the difference of setups heavily reflects on the techniques we use in the proofs.

In particular, if F is a convex set of objective lotteries, the model falls in the class of lottery dependent utility (see, e.g., Maccheroni, 2002, and the references therein).

While, when F is a set of simple (resp. continuous) acts, each u_l corresponds to a subjective expected utility with respect to a finitely additive (resp. countably additive) probability P_l. This time we are in the spirit of multiple priors models: for example, Choquet Expected Utility of Schmeidler (1989) and Maxmin Expected Utility of Gilboa and Schmeidler (1989) are particular cases of the proposed model when the family $\{F_l\}_{l\in L}$ consists of sets of comonotone and affinely related acts, respectively. In fact, many recent papers focus on specific cases of the model obtained here, and provide interesting interpretations on the derived family of probabilities. See, e.g. Nehring (2001), Ghirardato, Maccheroni, and Marinacci (2002), Kopylov (2002), Siniscalchi (2003). In particular, the latter work builds on a similar idea and looks for conditions ensuring the uniqueness of the subjective probability used to evaluate the expected utility of each act; furthermore, differently from us, the sets F_l are elicited from the preference.

2 A general representation result

Let F be a convex subset of a vector space, X a nonempty convex subset of F, $\{F_l\}_{l\in L}$ a family of convex subsets of F such that $F=\bigcup_{l\in L}F_l$ and $X\subseteq\bigcap_{l\in L}F_l$, and $\succsim$ a binary relation on F. As usual, we denote by $\succ$ and $\sim$ the asymmetric and the symmetric parts of $\succsim$. In the sequel we will make use of the following assumptions on $\succsim$.

Weak Order (WO): For all f_1 and f_2 in F: $f_1\succsim f_2$ or $f_2\succsim f_1$. For all f_1,f_2, and f_3 in F: if $f_1\succsim f_2$ and $f_2\succsim f_3$, then $f_1\succsim f_3$.

Local Independence (LI): For all $l\in L$, all f_1,f_2, and f_3 in F_l, and all α in $(0,1)$: $f_1\succsim f_2$ implies $\alpha f_1+(1-\alpha)f_3\succsim\alpha f_2+(1-\alpha)f_3$. When L is a singleton this property is the standard *Independence (I)*.

Local Continuity (LC): For all $l\in L$ and all f_1,f_2, and f_3 in F_l: if $f_1\succ f_2$ and $f_2\succ f_3$, then there exist α and β in $(0,1)$ such that $\alpha f_1+(1-\alpha)f_3\succ f_2$

and $f_2 \succ \beta f_1 + (1-\beta) f_3$. When L is a singleton this property is the standard *Continuity (C)*.

Boundedness (B): For all f in F: there exist $x_1, x_2 \in X$ such that $x_1 \succsim f$ and $f \succsim x_2$.

Quasiconcavity (Q): For all f_1 and f_2 in F and all α in $(0,1)$: $f_1 \sim f_2$ implies $\alpha f_1 + (1-\alpha) f_2 \succsim f_1$.

As suggested by Siniscalchi (2003), a natural way to elicit the sets F_l from the preference is to look for the maximal convex subsets of F on which it satisfies the standard assumptions of expected utility. Next theorem shows that the first four properties are necessary and sufficient to yield a piecewise affine representation of $\succsim$.

Theorem 1 *Given a binary relation $\succsim$ on F, the following conditions are equivalent:*

(i) $\succsim$ satisfies WO, LI, LC, and B.

(ii) There exists a family $\{u_l\}$ of affine functionals on F such that the functional

$$U(f) = u_l(f) \quad \textit{if } f \in F_l \tag{1}$$

represents $\succsim$ on F and $U(X) = U(F)$.

Moreover, U is unique up to positive affine transformations.

Ghirardato, Maccheroni, and Marinacci (2002), show that under suitable topological assumptions, the closed and convex hull of the family $\{u_l\}$ is the Clarke subdifferential of U.

Next we show that the quasiconcavity assumption Q implies concavity of the representation.

Corollary 1 *Let $\succsim$ be a binary relation represented by (1). Then, $\succsim$ satisfies Q if and only if $\{u_l\}$ can be chosen such that*

$$U(f) = \min_{l \in L} u_l(f)$$

for all $f \in F$.

It is easy to see that the assumptions WO, LI, LC, B, and Q are independent. Moreover, the Example on page 216 of Castagnoli and Maccheroni (2000) with $F = \mathbb{R}^2$ and $X = \{0\}$ shows that WO, LI, and LC are not sufficient to obtain a representation like (1). Further notice that $U(\alpha f + (1-\alpha)x) = \alpha U(f) + (1-\alpha) U(x)$ for all $\alpha \in [0,1]$, $f \in F$, and $x \in X$. We call this property *X-affinity*.

A special case of interest is the one in which only F and X are *a priori* given and, for all $f \in F - X$, F_f is the convex hull $\text{co}\{f, X\}$ of $\{f\}$ and X. In this case LI and LC can be restated with no explicit reference to the family F_f, moreover they can be replaced by

X-Independence (X-I): For all f_1, f_2 in F, all x in X, and all α in $(0,1)$: $f_1 \succsim f_2$ iff $\alpha f_1 + (1-\alpha)x \succsim \alpha f_2 + (1-\alpha)x$.

X-Continuity (X-C): For all $x_1, x_2 \in X$ and all f in F: if $x_1 \succ f$ and $f \succ x_2$, then there exist α and β in $(0,1)$ such that $\alpha x_1 + (1-\alpha)x_2 \succ f$ and $f \succ \beta x_1 + (1-\beta)x_2$.

The previous Theorem takes the following form.

Corollary 2 *Let $\succsim$ be a binary relation on F, and $F_f = \text{co}\{f, X\}$ for all $f \in F - X$. The following statements are equivalent:*

(i) *$\succsim$ satisfies WO, LI, LC, and B.*

(ii) *$\succsim$ satisfies WO, X-I, X-C, and B.*

(iii) *There exists an X-affine functional $U : F \to \mathbb{R}$ representing $\succsim$ and such that $U(X) = U(F)$.*

Moreover, U is unique up to positive affine transformations.

In this case, $\succsim$ satisfies Q iff there exists a family $\mathcal{U}$ of affine functionals on F, all of which are concordant on X, such that

$$U(f) = \min_{u \in \mathcal{U}} u(f).$$

We think that the above general results shed some light on the common traits of several well-known particular results in the literature. As an exemplification in the next section we apply them to a problem of choice under uncertainty. We are confident that they can be fruitfully employed to the study of different problems; e.g., decision models in which F is the convex set of all (closed and convex) sets of lotteries over a finite set Z of outcomes, and its elements are considered as menus of alternatives available to a decision maker (see, e.g., Dekel, Lipman, and Rustichini, 2001).

3 The Anscombe - Aumann setup

We now consider the special case in which F is a set of acts; more precisely, we focus on two possible settings.

The first one is the classical Anscombe - Aumann setup. S is a nonempty set of *states of the world*, Σ an algebra of subsets of S called *events*, X a convex set of *outcomes*. A *simple act* is just an X-valued, simple and Σ-measurable function; $F = F^s$ is the set of all simple acts. In this setting a *probability* on Σ is a finitely additive set function $P : \Sigma \to [0,1]$ such that $P(\emptyset) = 0$ and $P(S) = 1$.

The second one is a topological variation of the first. S is a compact metric set, Σ its Borel σ-field, and X a finite dimensional simplex. A *continuous act* is just an X-valued, continuous function; $F = F^c$ is the set of all continuous acts. In this setting a *probability* on Σ is a countably additive set function $P : \Sigma \to [0,1]$ such that $P(\emptyset) = 0$ and $P(S) = 1$.

For every $f_1, f_2 \in F$ and $\alpha \in [0,1]$ we denote by $\alpha f_1 + (1-\alpha) f_2$ the act in F which yields $\alpha f_1(s) + (1-\alpha) f_2(s) \in X$ for every $s \in S$. With a slight abuse of notation, we identify X with the set of all constant acts (thus making it a convex subset of F).

We will replace assumption B with the mildly stronger conditions:

Monotonicity (M): For all f_1 and f_2 in F: if $f_1(s) \succsim f_2(s)$ on S, then $f_1 \succsim f_2$.

Nondegeneracy (N): Not for all f_1 and f_2 in F, $f_1 \succsim f_2$.

Let $G \supseteq X$ be a subset of F, a functional $U : G \to \mathbb{R}$ is said to be *monotone* if $g_1(s) \succsim g_2(s)$ on S implies $U(g_1) \geq U(g_2)$; *automonotone* if $U(g_1(s)) \geq U(g_2(s))$ on S implies $U(g_1) \geq U(g_2)$ (that is, if U is monotone with respect to the pointwise dominance relation it induces on G). Next lemma is a little variation on the von Neumann - Morgenstern Theorem to yield a subjective probability result *à la* Anscombe and Aumann (1963). In particular, the lemma guarantees an expected utility representation for any preference $\succsim$ on G that satisfies WO, I, C, M, and N.

Lemma 1 *Let $G \supseteq X$ be a convex subset of F, $U : G \to \mathbb{R}$ a nonconstant, automonotone, affine functional, and u the restriction of U to X.*[1] *There exists a probability P on Σ such that*

$$U(g) = \int_S (u \circ g)\, dP$$

for all $g \in G$.

We are now ready to state the anticipated result.

Theorem 2 *Given a binary relation $\succsim$ on F, the following conditions are equivalent:*

(i) $\succsim$ satisfies WO, LI, LC, M, and N.

[1] More precisely: denoted by x_S the constant act taking value x for all $s \in S$, u is the function defined by $u(x) = U(x_S)$; the shorter expression we adopted derives from the identification of X with the set of all constant acts.

(ii) There exists a family $\{P_l\}_{l\in L}$ of probabilities on Σ, and an affine nonconstant function u on X, such that the functional

$$U(f) = \int_S (u \circ f)\, dP_l \quad \text{if } f \in F_l \tag{2}$$

represents $\succsim$ on F and it is monotone.

Moreover, U is unique up to positive affine transformations.

In the next corollary we consider the special case when the quasiconcavity axiom Q holds.

Corollary 3 *Let $\succsim$ be a binary relation represented by (2). Then, $\succsim$ satisfies Q if and only if $\{P_l\}_{l\in L}$ can be chosen such that*

$$U(f) = \min_{l\in L} \int_S (u \circ f)\, dP_l$$

for all $f \in F$.

The counterpart of Corollary 2 for $F = F^s$ is Theorem 1 of Gilboa and Schmeidler (1989), and we explicitly state it only in the case $F = F^c$. Here, the set of all probability measures is endowed with the weak* topology.

Corollary 4 *A binary relation $\succsim$ on F^c satisfies WO, X-I, X-C, M, N, and Q iff there exist an affine function $u : X \to \mathbb{R}$ and a compact and convex set $\mathcal{C}$ of probability measures, such that*

$$f \succsim g \Leftrightarrow \min_{P\in\mathcal{C}} \int_S (u \circ f)\, dP \geq \min_{P\in\mathcal{C}} \int_S (u \circ g)\, dP$$

for all $f, g \in F^c$. $\mathcal{C}$ is unique and u is unique up to a positive linear transformation.

Differently from the Gilboa and Schmeidler (1989) result, the set of priors $\mathcal{C}$ consists of countably additive probability measures. This way of obtaining countable additivity is alternative to that used by Marinacci, Maccheroni, Chateauneuf, and Tallon (2002); in fact, we add assumptions on the structure of the model rather than assumptions on the preference.

4 Proofs

Next Lemma is a minor variation on the Hahn - Banach Extension Theorem. Its proof is part of the one of Lemma 4 p. 829-830 in Maccheroni (2002).

Lemma 2 *Let $F \supseteq G \supseteq X$ be nonempty convex subsets of a vector space. If a functional $U : F \to \mathbb{R}$ is X-affine, concave, and $U_{|G}$ is affine, then there exists an affine functional $u : F \to \mathbb{R}$ such that $u \geq U$ and $u_{|G} = U_{|G}$.*

The following is a topological version of the previous one. We refer to Aliprantis and Border (1999) Chapter 5 for the basic notation and results on topological vector spaces' theory.

Lemma 3 *Let E be a vector space, E' be a total subspace of its algebraic dual, and K be a $\sigma(E',E)$-compact subset of E'. Set*

$$I(e) = \min_{e' \in K} \langle e, e' \rangle$$

for all $e \in E$. If I is affine on a convex subset C of E, there exists an extreme point e'_C of K such that $I_{|C} = e'_C$, i.e.

$$e'_C \in \operatorname{Argmin}_{e' \in K} \langle e, e' \rangle$$

for all $e \in C$.

Proof of Lemma 3. For all $e \in C$, $\operatorname{Argmin}_{e' \in K} \langle e, e' \rangle$ is a $\sigma(E',E)$-closed subset of K. By compactness of K, it is enough to show that $\bigcap_{j=1}^{n} \operatorname{Argmin}_{e' \in K} \langle e_j, e' \rangle \neq \emptyset$ for any $e_1, e_2, \ldots, e_n \in C$. Choose $w' \in \operatorname{Argmin}_{e' \in K} \left\langle \Sigma_{j=1}^{n} \frac{1}{n} e_j, e' \right\rangle$.

$$I\left(\Sigma_{j=1}^{n} e_j\right) = nI\left(\Sigma_{j=1}^{n} \frac{1}{n} e_j\right) = n\left\langle \Sigma_{j=1}^{n} \frac{1}{n} e_j, w' \right\rangle = \Sigma_{j=1}^{n} \left\langle e_j, w' \right\rangle$$

but, since I is affine on C

$$I\left(\Sigma_{j=1}^{n} e_j\right) = nI\left(\Sigma_{j=1}^{n} \frac{1}{n} e_j\right) = n\Sigma_{j=1}^{n} \frac{1}{n} I(e_j) = \Sigma_{j=1}^{n} \min_{e' \in K} \left\langle e_j, e' \right\rangle.$$

We can conclude that $w' \in K$ and

$$\Sigma_{j=1}^{n} \left\langle e_j, w' \right\rangle = \Sigma_{j=1}^{n} \min_{e' \in K} \left\langle e_j, e' \right\rangle.$$

Hence

$$\left\langle e_j, w' \right\rangle = \min_{e' \in K} \left\langle e_j, e' \right\rangle$$

for all $j = 1, 2, \ldots, n$, that is $w' \in \bigcap_{j=1}^{n} \operatorname{Argmin}_{e' \in K} \left\langle e_j, e' \right\rangle$.

Moreover, $\bigcap_{e \in C} \operatorname{Argmin}_{e' \in K} \langle e, e' \rangle$ is a nonempty intersection of compact extreme sets, hence it is a compact extreme set, and it contains an extreme point. **Q.E.D.**

Proof of Theorem 1 and Corollary 1.[2] By the von Neumann - Morgenstern Theorem for all $l \in L$ there exists an affine functional

$$u_l : F_l \to \mathbb{R}$$

[2] The proofs are not separated to avoid duplicate notation.

representing $\succsim$ on F_l. We still denote by u_l an arbitrarily fixed affine extension of u_l to F. Since $u_{l|X}$ is an affine representation of $\succsim$ on X, it is unique up to positive affine transformations. Fix arbitrarily $m \in L$ and set $u = u_{m|X}$. For all $l \in L$ choose u_l so that $u_{l|X} = u$.

By B, for all $f \in F$ there exist $x_1, x_2 \in X$ such that $x_1 \succsim f \succsim x_2$. Therefore $u(x_1) = u_l(x_1) \geq u_l(f) \geq u_l(x_2) = u(x_2)$, for all $l \in L$ such that $f \in F_l$, and there exists $\alpha \in [0,1]$ such that

$$\begin{aligned} u_l(f) &= \alpha u(x_1) + (1-\alpha) u(x_2) \\ &= u(\alpha x_1 + (1-\alpha) x_2) \\ &= u_l(\alpha x_1 + (1-\alpha) x_2), \end{aligned}$$

therefore $u_l(f)$ does not depend on the choice of $l \in L$ such that $f \in F_l$. Moreover, the argument above shows that there exists $x_f \in X$ (i.e. $\alpha x_1 + (1-\alpha) x_2$) such that $x_f \sim f$ and

$$u_l(f) = u(x_f)$$

for all $l \in L$ such that $f \in F_l$.

We set

$$U(f) = u_l(f) \quad \text{if } f \in F_l.$$

What precedes guarantees that U is well defined, and $U(f) = u(x_f) = U(x_f)$ implies $U(F) = U(X)$. For all $f_1, f_2 \in F$, let $f_i \sim x_i \in X$ to obtain

$$f_1 \succsim f_2 \Leftrightarrow x_1 \succsim x_2 \Leftrightarrow u(x_1) \geq u(x_2) \Leftrightarrow U(f_1) \geq U(f_2).$$

If $U' : F \to \mathbb{R}$ is affine on F_l for all $l \in L$ and represents $\succsim$, then $u' = U'_{|X} = au + b$ for some $a > 0$ and $b \in \mathbb{R}$; for all $f \in F$, let $f \sim x_f \in X$ to obtain

$$U'(f) = u'(x_f) = au(x_f) + b = aU(f) + b.$$

This concludes the proof of Theorem 1.

For any $\alpha \in [0,1]$, $f \in F$, and $x \in X$, choose $l \in L$ such that $f \in F_l$ to obtain

$$\begin{aligned} U(\alpha f + (1-\alpha) x) &= u_l(\alpha f + (1-\alpha) x) \\ &= \alpha u_l(f) + (1-\alpha) u_l(x) \\ &= \alpha U(f) + (1-\alpha) U(x), \end{aligned}$$

this shows that U is X-affine.

Next we prove Corollary 1. If U is constant, the result is trivial. If U is not constant, there exist $f_1, f_2 \in F$ such that $f_1 \succ f_2$ and, by B, there exist $x_1^*, x_{-1}^* \in X$ such that $x_1^* \succ x_{-1}^*$. W.l.o.g. assume $x_{-1}^* = -x_1^*$ (so that $0 \in X$) and $U(x_1^*) = 1$, $U\left(x_{-1}^*\right) = -1$, whence $U(0) = U\left(\frac{1}{2}x_1^* + \frac{1}{2}x_{-1}^*\right) = 0$. Then U is positively homogeneous. The (unique) positively homogeneous extension of U to the convex cone H generated by F is the functional defined by

$$V(\gamma f) = \gamma U(f)$$

if $f \in F$ and $\gamma > 0$. Let $h \in H$ and y in the convex cone Y generated by X, there exist $\gamma > 0$, $f \in F$ and $x \in X$ such that $h = \gamma f$ and $y = \gamma x$, whence

$$\begin{aligned} \frac{1}{2}V(h+y) &= \frac{1}{2}V(\gamma(f+x)) \\ &= \gamma V\left(\frac{1}{2}(f+x)\right) \\ &= \gamma U\left(\frac{1}{2}f+\frac{1}{2}x\right) \\ &= \gamma\left(\frac{1}{2}V(f)+\frac{1}{2}V(x)\right) \\ &= \frac{1}{2}(V(h)+V(y)), \end{aligned}$$

that is $V(h+y) = V(h)+V(y)$.

Let $h_1, h_2 \in H$; there exist $\gamma > 0$, $f_1, f_2 \in F$ such that $h_i = \gamma f_i$. If $V(h_1) = V(h_2)$, $U(f_1) = V(f_1) = V(f_2) = U(f_2)$, so that $f_1 \sim f_2$ and $U\left(\frac{1}{2}f_1 + \frac{1}{2}f_2\right) \geq U(f_1) = \frac{1}{2}U(f_1)+\frac{1}{2}U(f_2)$, that is $V(h_1+h_2) \geq V(h_1)+V(h_2)$. Else if $V(h_1) > V(h_2)$, there exists $y \in Y$ such that $V(y) = V(h_1) - V(h_2)$ (take $(V(h_1)-V(h_2))x_1^*$), then

$$\begin{aligned} V(h_1+h_2)+V(y) &= V(h_1+h_2+y) \\ &\geq V(h_1)+V(h_2+y) \\ &= V(h_1)+V(h_2)+V(y). \end{aligned}$$

That is, V is superlinear and U is concave. Now using Lemma 2 for each F_l we can choose v_l such that $v_l : F \to \mathbb{R}$ is affine, $v_l \geq U$ and $v_{l|F_l} = U_{|F_l}$. Replace the u_l chosen at the beginning of the proof with v_l to obtain

$$U(f) = v_l(f) = \min_{i \in L} v_i(f)$$

if $f \in F_l$. The rest is trivial. **Q.E.D.**

Given Theorem 1 and Corollary 1, the **proof of Corollary 2** is a long, simple exercise.

We denote by $B_0(S,\Sigma)$ the vector space of all real valued, simple and Σ-measurable functions, endowed with the supnorm topology. If S is a compact metric set, we denote by $C(S)$ the vector space of all real valued, continuous functions, endowed with the supnorm topology. It is well known that the topological dual of $B_0(S,\Sigma)$ (resp. $C(S)$) is the vector space $ba(S,\Sigma)$ of all bounded, finitely additive set functions on Σ (resp. the vector space $ca(S)$ of all countably additive set functions on Σ): the duality being

$$\langle \varphi, \mu \rangle = \int_S \varphi d\mu$$

for all $\varphi \in B_0(S,\Sigma)$ and $\mu \in ba(S,\Sigma)$ (resp. $\varphi \in C(S)$ and $\mu \in ca(S)$). If $k \in \mathbb{R}$, the constant element of $B_0(S,\Sigma)$ or $C(S)$ taking value k on S will be denoted again by k. A functional I on a subset of $B_0(S,\Sigma)$ or $C(S)$ is *monotone* if $\varphi_1 \geq \varphi_2$ implies $I(\varphi_1) \geq I(\varphi_2)$. A monotone linear functional I on $B_0(S,\Sigma)$ or $C(S)$ corresponds to a positive set function μ.

Proof of Lemma 1. Let $u = U_{|X}$; obviously u is affine, (and continuous if $F = F^c$). For all $g \in G$, let $\overline{x} \in g(S)$ be such that $u(\overline{x}) \geq u(g(s))$ for all $s \in S$ and $\underline{x} \in g(S)$ be such that $u(\underline{x}) \leq u(g(s))$ for all $s \in S$. The existence of such $\overline{x}$ and $\underline{x}$ descends from the finiteness of $g(S)$ if $F = F^s$, from the continuity of g and u if $F = F^c$. Then $U(\underline{x}) \leq U(g) \leq U(\overline{x})$, and there exists $x_g \in X$ such that $U(x_g) = U(g)$. Hence $U(G) = U(X)$ and there exists $x_*, x^* \in \operatorname{int} U(X)$ with $U(x_*) < U(x^*)$. Assume first $-U(x_*) = U(x^*) = 1$. Automonotonicity of U yields that $g_1, g_2 \in G$ and $u \circ g_1 = u \circ g_2$ imply $U(g_1) = U(g_2)$. It is easy to see that $\Phi = \{u \circ g : g \in G\}$ is a convex subset of $B_0(S,\Sigma)$ or $C(S)$ containing the constant functions 1 and -1.

Define $I : \Phi \to \mathbb{R}$ by

$$I(\varphi) = U(g)$$

if $\varphi = u \circ g$. I is monotone, affine, $I(0) = 0$ and $I(1) = 1$. It is routine to extend I to the vector subspace $\langle \Phi \rangle$ of $B_0(S,\Sigma)$ or $C(S)$ generated by Φ and obtain a linear, monotone functional $\hat{I} : \langle \Phi \rangle \to \mathbb{R}$ such that $\hat{I}(0) = 0$ and $\hat{I}(1) = 1$. A classical extension result of Kantorovich (see, e.g., Aliprantis and Border, 1999, Lemma 7.31) guarantees that there exists a linear, monotone extension $\tilde{I}$ of $\hat{I}$ to the whole $B_0(S,\Sigma)$ or $C(S)$. We can conclude that there exists a probability P on Σ such that

$$U(g) = I(u \circ g) = \int_S (u \circ g) \, dP$$

for all $g \in G$.

Finally, if it is not the case that $-U(x_*) = U(x^*) = 1$, there exist $a > 0$ and $b \in \mathbb{R}$ such that $-(aU(x_*) + b) = (aU(x^*) + b) = 1$, and the proposed technique yields

$$aU(g) + b = \int_S (a(u \circ g) + b) \, dP$$

for all $g \in G$, as wanted. **Q.E.D.**

Proof of Theorem 2 and Corollary 3.[3] M implies B. If $F = F^s$, for any act f take $\overline{x} \in f(S)$ such that $\overline{x} \succsim f(s)$ for all $s \in S$ and $\underline{x} \in f(S)$ such that $f(s) \succsim \underline{x}$ for all $s \in S$ to obtain $\overline{x} \succsim f$ and $f \succsim \underline{x}$. If $F = F^c$, let $v : X \to \mathbb{R}$ be an affine function that represents $\succsim$ on X; for any act f, there exists $\underline{s}$ and $\overline{s}$ such that $v(f(\overline{s})) \geq v(f(s)) \geq v(f(\underline{s}))$ for all $s \in S$, then M guarantees that $f(\overline{s}) \succsim f \succsim f(\underline{s})$. By Theorem 1 there exists a functional $U : F \to \mathbb{R}$, affine on F_l for all $l \in L$, that represents $\succsim$ (and for all $f \in F$ there exists $x_f \in X$ such that $x_f \sim f$).

[3] The proofs are not separated to avoid duplicate notation.

M also implies that U is automonotone on F (*a fortiori* on F_l for all $l \in L$). In fact, $U(f_1(s)) \geq U(f_2(s))$ on S implies $f_1(s) \succsim f_2(s)$ on S and $f_1 \succsim f_2$, whence $U(f_1) \geq U(f_2)$. M and N imply that U is nonconstant on F_l for all $l \in L$ (just take $f_1^* \succ f_{-1}^*$, and $x_1^*, x_{-1}^* \in X$ with $x_i^* \sim f_i$ to have $U(x_1^*) > U(x_{-1}^*)$). Apply Lemma 1 to F_l for each $l \in L$ to obtain a family $\{P_l\}_{l \in L}$ of probabilities on Σ such that

$$U(f) = \int_S (u \circ f)\, dP_l \quad \text{if } f \in F_l,$$

where $u : X \to \mathbb{R}$ is the restriction of U to X. This proves Theorem 2.

Next we prove Corollary 3. Assuming Q holds, then U is concave.

If $F = F^s$, w.l.o.g. $u(X) \supseteq [-1,1]$, and $\{u \circ f : f \in F\}$ is the set $B_0(S, \Sigma, u(X))$ of simple, Σ measurable functions from S to $u(X)$.

Else if $F = F^c$, w.l.o.g. $u(X) = [-1,1]$, and $\{u \circ f : f \in F\}$ is the set $C(S, u(X))$ of continuous functions from S to $u(X)$.[4]

For all $\varphi \in B_0(S, \Sigma, u(X))$ or $C(S, u(X))$, set

$$I(\varphi) = U(f)$$

if $\varphi = u \circ f$. I is monotone, $u(X)$-affine, concave, $I(0) = 0$ and $I(1) = 1$. Therefore, its positive homogeneous extension $\hat{I}$ to $B_0(S, \Sigma)$ or $C(S)$ is monotone, superlinear, and such that $\hat{I}(\varphi + k) = \hat{I}(\varphi) + k$ for all $\varphi \in B_0(S, \Sigma)$ or $C(S)$ and all $k \in \mathbb{R}$. Moreover, being bounded on $B_0(S, \Sigma, [-1,1])$ or $C(S, [-1,1])$, $\hat{I}$ is continuous in the supnorm. Standard convex analysis results guarantee that there exists a unique convex and weak* compact set $\mathcal{C}$ of probabilities such that

$$\hat{I}(\varphi) = \min_{P \in \mathcal{C}} \int_S \varphi dP$$

(just take as $\mathcal{C}$ the superdifferential of $\hat{I}$ at 0). The functional $\hat{I}$ is affine on the convex set $\Phi_l = \{u \circ f : f \in F_l\}$ for all $l \in L$. In fact, for all $l \in L$ and all $\varphi_i = u \circ f_i$ with $f_i \in F_l$, and $\alpha \in [0,1]$ we have

$$\begin{aligned} \hat{I}(\alpha\varphi_1 + (1-\alpha)\varphi_1) &= I(u \circ (\alpha f_1 + (1-\alpha) f_2)) \\ &= U(\alpha f_1 + (1-\alpha) f_2) \\ &= \alpha U(f_1) + (1-\alpha) U(f_2) \\ &= \alpha \hat{I}(\varphi_1) + (1-\alpha)\hat{I}(\varphi_2). \end{aligned}$$

By Lemma 3, there exist $P_l' \in \mathcal{C}$ such that

$$\hat{I}(\varphi) = \int_S \varphi dP_l'$$

[4] Let $x_1^* \in u^{-1}(1)$ and $x_{-1}^* \in u^{-1}(-1)$. The restriction ν of u to $[x_{-1}^*, x_1^*]$ is an homeomorphism between $[x_{-1}^*, x_1^*]$ and $[-1,1]$; so if $\varphi : S \to [-1,1]$ is continuous, $f = \nu^{-1} \circ \varphi : S \to [x_{-1}^*, x_1^*] \subseteq X$ is a continuous act such that

$$u(f(s)) = u(\nu^{-1}(\varphi(s))) = \nu(\nu^{-1}(\varphi(s))) = \varphi(s).$$

for all $\varphi \in \Phi_l$. Therefore for all $l \in L$ and all $f \in F_l$

$$U(f) = I(u \circ f) = \min_{P \in C} \int_S (u \circ f)\, dP = \int_S (u \circ f)\, dP'_l = \min_{m \in L} \int_S (u \circ f)\, dP'_m.$$

The rest is trivial. **Q.E.D.**

The **proof of Corollary 4** is immediate.

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