Combining Belief Functions Issued from Dependent Sources

MARCO E.G.V. CATTANEO
ETH Zürich, Switzerland

Abstract

Dempster's rule for combining two belief functions assumes the independence of the sources of information. If this assumption is questionable, I suggest to use the least specific combination minimizing the conflict among the ones allowed by a simple generalization of Dempster's rule. This increases the monotonicity of the reasoning and helps us to manage situations of dependence. Some properties of this combination rule and its usefulness in a generalization of Bayes' theorem are then considered.

Keywords

belief functions, propositional logic, combination, Dempster's rule, independence, conflict, monotonicity, nonspecificity, idempotency, associativity, Bayes' theorem

1 Introduction

In the theory of belief functions, Dempster's rule allows us to pool the information issued from several sources, if we assume that these are independent. In his original work [2], Dempster based the independence concept on the usual statistical one and underlined the vagueness of its real world meaning. Shafer reinterpreted Dempster's work and in his monograph [8] defined a belief function without assuming an underlying probability space, making so the independence assumption even more problematic.

In probability theory, the independence concept refers to classes of events or to random variables, with respect to a single probability distribution (this kind of independence for belief functions is studied for instance in Ben Yaghlane, Smets and Mellouli [1]). On the contrary, the concept considered here refers to several sources of information issuing several belief functions over the same frame of discernment. The assumption of the independence of the sources can be justified only by analogies with other situations in which this assumption proved to be sensible (cf. Smets [10]).

Following Dubois and Prade [3], I consider a generalization of Dempster's rule which allows the sources of information to be dependent. This general rule just assigns to a pair of belief functions a set of possible combinations, compelling us to make a choice. If the independence of the sources of information is doubtful (that is, we cannot adequately justify its assumption), I suggest to choose the least specific combination minimizing the conflict. This increases the monotonicity of the reasoning (in particular, complete monotonicity is assured if it does not entail incoherence) and helps us to manage situations of dependence (in particular, idempotency is assured).

2 Setting and Notation

It is assumed that the reader has a basic knowledge of the Dempster-Shafer theory and of classical propositional logic (refer for instance to Shafer [8] and to Epstein [4], respectively).

Let $\mathcal{U}$ be a finite set of propositional variables, which represents the topic considered. $\mathcal{L}_{\mathcal{U}}$ denotes the language of propositional logic built over the alphabet $\mathcal{U} \cup \{\top, \neg, \vee, \wedge, \rightarrow\}$, where $\top$ is the tautology. $V_{\mathcal{U}}$ denotes the set of (classical) valuations of $\mathcal{L}_{\mathcal{U}}$, i.e. the consistent assignments $v : \mathcal{L}_{\mathcal{U}} \longrightarrow \{t, f\}$ of truth values to the formulas of $\mathcal{L}_{\mathcal{U}}$ (thus $|V_{\mathcal{U}}| = 2^{|\mathcal{U}|}$). The mapping

$$\begin{array}{rrcl} T_{\mathcal{U}}: & \mathcal{L}_{\mathcal{U}} & \longrightarrow & 2^{V_{\mathcal{U}}} \\ & \varphi & \longmapsto & \{v \in V_{\mathcal{U}} : v(\varphi) = t\} \end{array}$$

assigns to each formula of $\mathcal{L}_{\mathcal{U}}$ the set of its models, i.e. the valuations for which the formula is true.[1]

Definition 1 *A basic belief assignment (bba) is a function*

$$m : 2^{V_{\mathcal{U}}} \longrightarrow [0,1] \text{ such that } m(\emptyset) = 0 \text{ and } \sum_{A \subseteq V_{\mathcal{U}}} m(A) = 1.^2$$

$\mathcal{M}_{\mathcal{U}}$ is the set of bbas on $2^{V_{\mathcal{U}}}$.

The belief and the plausibility about $\mathcal{U}$ with bba m are the functions

$$\begin{array}{rrcl} bel: & \mathcal{L}_{\mathcal{U}} & \longrightarrow & [0,1] \\ & \varphi & \longmapsto & \sum_{A \subseteq T_{\mathcal{U}}(\varphi)} m(A), \\ pl: & \mathcal{L}_{\mathcal{U}} & \longrightarrow & [0,1] \\ & \varphi & \longmapsto & \sum_{A \cap T_{\mathcal{U}}(\varphi) \neq \emptyset} m(A). \end{array}$$

[1] $T_{\mathcal{U}}$ is not injective ($\mathcal{L}_{\mathcal{U}}$ is redundant) but it is surjective ($\mathcal{L}_{\mathcal{U}}$ is sufficient).

[2] The beliefs are normalized, since the "open-world assumption" (see for instance Smets [9]) does not make sense in the setting of classical propositional logic: a formula and its negation cannot both be false.

Consider two finite sets of propositional variables $\mathcal{U} \subseteq \mathcal{V}$. If *bel* is a belief about $\mathcal{V}$, the belief $bel\downarrow_{\mathcal{U}}$ about $\mathcal{U}$ is the restriction of *bel* to $\mathcal{L}_{\mathcal{U}}$. If *bel* is a belief about $\mathcal{U}$, the belief $bel\uparrow^{\mathcal{V}}$ about $\mathcal{V}$ is the vacuous extension of *bel* to $\mathcal{L}_{\mathcal{V}}$, i.e. the minimal belief about $\mathcal{V}$ whose restriction to $\mathcal{L}_{\mathcal{U}}$ is *bel* (where minimal means that if bel' is a belief about $\mathcal{V}$ satisfying $bel'\downarrow_{\mathcal{U}}= bel$, then $bel\uparrow^{\mathcal{V}}\leq bel'$).[3]

Definition 2 *A joint belief assignment (jba) with marginal bbas $m_1, m_2 \in \mathcal{M}_{\mathcal{U}}$ is a function*

$$\begin{aligned} &\underline{m} : 2^{V_{\mathcal{U}}} \times 2^{V_{\mathcal{U}}} \longrightarrow [0,1] \text{ such that} \\ &\sum_{B \subseteq V_{\mathcal{U}}} \underline{m}(A,B) = m_1(A) \text{ for all } A \subseteq V_{\mathcal{U}} \text{ and} \\ &\sum_{A \subseteq V_{\mathcal{U}}} \underline{m}(A,B) = m_2(B) \text{ for all } B \subseteq V_{\mathcal{U}}. \end{aligned}$$

$\underline{\mathcal{M}}_{\mathcal{U}}^{m_1,m_2}$ *is the set of jbas with marginal bbas $m_1, m_2 \in \mathcal{M}_{\mathcal{U}}$.*

The conflict of a jba $\underline{m}$ is the quantity

$$c(\underline{m}) = \sum_{A \cap B = \emptyset} \underline{m}(A,B).$$

For any $m_1, m_2 \in \mathcal{M}_{\mathcal{U}}$, the function $\underline{m}_D$ on $2^{V_{\mathcal{U}}} \times 2^{V_{\mathcal{U}}}$ defined by

$$\underline{m}_D(A,B) = m_1(A)\, m_2(B)$$

is a jba with marginal bbas m_1 and m_2 (it is the jba which corresponds to the independence assumption). Thus $\underline{\mathcal{M}}_{\mathcal{U}}^{m_1,m_2}$ cannot be empty.

In the following, bel_1 and bel_2 will denote two beliefs about $\mathcal{U}$ with bbas m_1 and m_2, respectively (and pl_1 and pl_2 will denote the respective plausibilities). If $\underline{m} \in \underline{\mathcal{M}}_{\mathcal{U}}^{m_1,m_2}$ with $c(\underline{m}) < 1$, the function m on $2^{V_{\mathcal{U}}}$ defined by $m(\emptyset) = 0$ and

$$m(A) = \frac{1}{1 - c(\underline{m})} \sum_{B \cap C = A} \underline{m}(B,C) \text{ if } A \neq \emptyset$$

is a bba. The belief about $\mathcal{U}$ with bba m is called combination of bel_1 and bel_2 with respect to $\underline{m}$, and is denoted by $bel_1 \otimes_{\underline{m}} bel_2$. The rule $\otimes$ generalizes Dempster's one $\oplus$, since the latter is the combination with respect to the particular jba $\underline{m}_D$, or symbolically $\oplus = \otimes_{\underline{m}_D}$.

3 Monotonicity and Conflict

A reasoning process is called monotonic if the acquisition of new information does not compel us to give up some of our beliefs; otherwise it is called non-monotonic. In the Dempster-Shafer theory, the reasoning process consists in the

[3] If m is the bba associated with *bel*, then the bba associated with $bel\uparrow^{\mathcal{V}}$ is the function m' on $2^{V_{\mathcal{V}}}$ defined by $m'(T_{\mathcal{V}}(\varphi)) = m(T_{\mathcal{U}}(\varphi))$ for all $\varphi \in \mathcal{L}_{\mathcal{U}}$, and $m'(A) = 0$ if $A \notin T_{\mathcal{V}}(\mathcal{L}_{\mathcal{U}})$.

combination of beliefs. That is, the reasoning would be monotonic only if

$$bel_1 \otimes_{\underline{m}} bel_2 \geq \max\left(bel_1, bel_2\right),$$

which does not always hold (cf. Yager [12]). Proposition 1 gives the best possible lower bound for $bel_1 \otimes_{\underline{m}} bel_2(\varphi)$ based only on the knowledge of $bel_1(\varphi)$, $bel_2(\varphi)$ and $c(\underline{m})$.

Proposition 1 *If $\underline{m} \in \underline{\mathcal{M}}_{\mathcal{U}}^{m_1,m_2}$ with $c(\underline{m}) < 1$, and $\varphi \in \mathcal{L}_{\mathcal{U}}$, then*

$$bel_1 \otimes_{\underline{m}} bel_2(\varphi) \geq \max\left(\frac{bel_1(\varphi) - c(\underline{m})}{1 - c(\underline{m})}, \frac{bel_2(\varphi) - c(\underline{m})}{1 - c(\underline{m})}, 0\right).$$

Proof. $$(1 - c(\underline{m}))\, bel_1 \otimes_{\underline{m}} bel_2(\varphi) = \sum_{\emptyset \neq (A \cap B) \subseteq T_{\mathcal{U}}(\varphi)} \underline{m}(A,B) \geq$$
$$\geq \sum_{A \subseteq T_{\mathcal{U}}(\varphi)} \sum_{B \subseteq V_{\mathcal{U}}} \underline{m}(A,B) - \sum_{A \cap B = \emptyset} \underline{m}(A,B) = bel_1(\varphi) - c(\underline{m}).$$

Similarly, $(1 - c(\underline{m}))\, bel_1 \otimes_{\underline{m}} bel_2(\varphi) \geq bel_2(\varphi) - c(\underline{m})$. □

From Proposition 1 it follows that if $\underline{m}$ has no conflict (i.e. $c(\underline{m}) = 0$), then we have monotonicity. But if $\underline{m}$ has some conflict (i.e. $c(\underline{m}) > 0$), then the monotonicity is assured only for the formulas φ such that $\max\left(bel_1(\varphi), bel_2(\varphi)\right) = 1$. In general we can affirm that the more $\underline{m}$ has conflict, the more we have nonmonotonicity.

The monotonicity is admissible only if there is a belief bel about $\mathcal{U}$ with $bel \geq \max\left(bel_1, bel_2\right)$. If there is a formula φ with $bel_1(\varphi) > pl_2(\varphi)$,[4] then the monotonicity is not admissible, since $bel \geq \max\left(bel_1, bel_2\right)$ implies that

$$bel(\top) \geq bel(\varphi) + bel(\neg\varphi) \geq bel_1(\varphi) + bel_2(\neg\varphi) > 1.$$

Proposition 2 assures that if the monotonicity is admissible, then it is feasible (that is, there is a jba without conflict).

Proposition 2

$$\min_{\underline{m} \in \underline{\mathcal{M}}_{\mathcal{U}}^{m_1,m_2}} c(\underline{m}) = \max_{\varphi \in \mathcal{L}_{\mathcal{U}}} \left(bel_1(\varphi) - pl_2(\varphi)\right).$$

Proof. Let $\underline{m}$ be a jba minimizing the conflict (such a jba certainly exists since $\underline{\mathcal{M}}_{\mathcal{U}}^{m_1,m_2} \subset \mathbb{R}^{2^{2|V_{\mathcal{U}}|}}$ is compact and not empty).

If $A_1, A_2, B_1, B_2 \subseteq V_{\mathcal{U}}$ with $A_1 \cap B_1 = \emptyset$, $A_1 \cap B_2 \neq \emptyset$, $A_1 \neq A_2$, $\underline{m}(A_1, B_1) > 0$ and $\underline{m}(A_2, B_2) > 0$, then $A_2 \cap B_1 = \emptyset$ and $A_2 \cap B_2 \neq \emptyset$, and without loss of generality we may assume that $\underline{m}(A_2, B_1) > 0$.

[4] Notice that $bel_2(\psi) - pl_1(\psi) = bel_1(\varphi) - pl_2(\varphi)$ with $\varphi = \neg\psi$.

To prove this, consider the function $\underline{m}'$ on $2^{V_{\mathcal{U}}} \times 2^{V_{\mathcal{U}}}$ defined by

$$\underline{m}'(A,B) = \begin{cases} \underline{m}(A,B) - \varepsilon & \text{if } (A,B) \in \{(A_1,B_1),(A_2,B_2)\}, \\ \underline{m}(A,B) + \varepsilon & \text{if } (A,B) \in \{(A_1,B_2),(A_2,B_1)\}, \\ \underline{m}(A,B) & \text{otherwise,} \end{cases}$$

for an ε such that $0 < \varepsilon < \min(\underline{m}(A_1,B_1), \underline{m}(A_2,B_2))$. It is easily verified that $\underline{m}' \in \underline{\mathcal{M}}_{\mathcal{U}}^{m_1,m_2}$ and $c(\underline{m}') \leq c(\underline{m})$, with equality only if $A_2 \cap B_1 = \emptyset$ and $A_2 \cap B_2 \neq \emptyset$.

Let $\mathcal{A} = \{A \subseteq V_{\mathcal{U}} : \exists\, B \subseteq V_{\mathcal{U}}\ A \cap B = \emptyset, \underline{m}(A,B) > 0\}$ and $\underline{\mathcal{A}} = \bigcup_{A \in \mathcal{A}} A$.

If $B \cap \underline{\mathcal{A}} \neq \emptyset$, then $m_2(B) = \sum_{A \in \mathcal{A}, A \cap B \neq \emptyset} \underline{m}(A,B)$.

This can be proven as follows. Since $B \cap \underline{\mathcal{A}} \neq \emptyset$, there is an $A_1 \in \mathcal{A}$ with $A_1 \cap B \neq \emptyset$. Since $A_1 \in \mathcal{A}$, there is a $B_1 \subseteq V_{\mathcal{U}}$ with $A_1 \cap B_1 = \emptyset$ and $\underline{m}(A_1,B_1) > 0$. If $A_2 \subseteq V_{\mathcal{U}}$ with $A_1 \neq A_2$ and $\underline{m}(A_2,B) > 0$, then we are in the situation considered above (with $B_2 = B$). Therefore $A_2 \in \mathcal{A}$ (since $A_2 \cap B_1 = \emptyset$ and $\underline{m}(A_2,B_1) > 0$) and $A_2 \cap B \neq \emptyset$. Thus $\underline{m}(A,B) > 0$ implies $A \in \mathcal{A}$ and $A \cap B \neq \emptyset$.

Let $\varphi \in \mathcal{L}_{\mathcal{U}}$ with $T_{\mathcal{U}}(\varphi) = \underline{\mathcal{A}}$. Then

$$c(\underline{m}) = \sum_{A \in \mathcal{A}, A \cap B = \emptyset} \underline{m}(A,B) = \sum_{A \in \mathcal{A}} m_1(A) - \sum_{A \in \mathcal{A}, A \cap B \neq \emptyset} \underline{m}(A,B) =$$
$$= \sum_{A \in \mathcal{A}} m_1(A) - \sum_{B \cap \underline{\mathcal{A}} \neq \emptyset} m_2(B) \leq bel_1(\varphi) - pl_2(\varphi).$$

On the other hand, for any $\psi \in \mathcal{L}_{\mathcal{U}}$ (let $C = T_{\mathcal{U}}(\psi)$ and $\overline{C} = V_{\mathcal{U}} \backslash C$),

$$c(\underline{m}) \geq \sum_{A \subseteq C, B \subseteq \overline{C}} \underline{m}(A,B) = \sum_{A \subseteq C} m_1(A) - \sum_{A \subseteq C, B \not\subseteq \overline{C}} \underline{m}(A,B) \geq$$
$$\geq \sum_{A \subseteq C} m_1(A) - \sum_{B \not\subseteq \overline{C}} m_2(B) = bel_1(\psi) - pl_2(\psi)$$

□

Let $c_{\min}^{m_1,m_2}$ denote the value of $\min_{\underline{m} \in \underline{\mathcal{M}}_{\mathcal{U}}^{m_1,m_2}} c(\underline{m})$, and let bel_1 and bel_2 be called compatible if $bel_1 \leq pl_2$. Proposition 2 enables us to determine $c_{\min}^{m_1,m_2}$ and to prove Corollary 1.

Corollary 1 *The following assertions are equivalent.*

- *The monotonicity of the combination of bel_1 and bel_2 is admissible.*
- *bel_1 and bel_2 are compatible.*
- *$c_{\min}^{m_1,m_2}=0$.*

4 The Choice of a Combination Rule

The only case in which the marginal bbas uniquely determine the jba is the conditioning of a belief. The conditioning on $\varphi \in \mathcal{L}_{\mathcal{U}}$ of a belief *bel* about $\mathcal{U}$ is the result of its combination with $bel_{\mathcal{U}}^{\varphi}$, where $bel_{\mathcal{U}}^{\varphi}$ denotes the minimal belief about $\mathcal{U}$ assigning the value 1 to the formula φ.[5] It is easily verified that if one of the two beliefs which have to be combined has the form $bel_{\mathcal{U}}^{\varphi}$, then the jba is unique (i.e. $\underline{\mathcal{M}}_{\mathcal{U}}^{m_1,m_2} = \{\underline{m}_D\}$). Thus in the case of conditioning the general rule $\otimes$ reduces itself to Dempster's one.

Generally, in order to combine two beliefs bel_1 and bel_2 about $\mathcal{U}$, we must choose a jba $\underline{m} \in \underline{\mathcal{M}}_{\mathcal{U}}^{m_1,m_2}$. Sometimes we can analyse in detail the situation and base our choice on specific assumptions about the nature of the dependence of the sources of information, but usually we can at most assume their independence. Thus there is little loss of generality in considering only the two usual cases: the one in which the independence is assumed, and the one in which nothing is assumed about the sources. In both cases we need a combination rule; that is, we need an operator $\star$ assigning to every pair of bbas $m_1, m_2 \in \mathcal{M}_{\mathcal{U}}$ a jba $m_1 \star m_2 \in \underline{\mathcal{M}}_{\mathcal{U}}^{m_1,m_2}$, for any finite set of propositional variables $\mathcal{U}$. Such an operator can be sensible only if it satisfies the following basic requirements (the first two make the combination rule independent of the particular logical formalization, whereas the third one is a technical necessity).

- The influence of $\mathcal{U}$ on $\star$ must be limited to the cardinality of $V_{\mathcal{U}}$. That is, if $\mathcal{V}$ is a set of propositional variables and $f : V_{\mathcal{V}} \longrightarrow V_{\mathcal{U}}$ is a bijection, then

$$(m_1 \circ f) \star (m_2 \circ f)(A,B) = m_1 \star m_2 (f(A), f(B)) \text{ for all } A,B \subseteq V_{\mathcal{V}}.$$

- The operator $\star$ must be "equivariant" with respect to the vacuous extensions. That is, if $\mathcal{V}$ is a finite set of propositional variables with $\mathcal{U} \subseteq \mathcal{V}$ and m_1', m_2' are the bbas associated with $bel_1 \uparrow^{\mathcal{V}}$ and $bel_2 \uparrow^{\mathcal{V}}$, respectively, then

$$m_1' \star m_2' (T_{\mathcal{V}}(\varphi), T_{\mathcal{V}}(\psi)) = m_1 \star m_2 (T_{\mathcal{U}}(\varphi), T_{\mathcal{U}}(\psi)) \text{ for all } \varphi, \psi \in \mathcal{L}_{\mathcal{U}}.$$

- The combination with respect to $m_1 \star m_2$ must be defined as often as possible. That is, if $c_{\min}^{m_1,m_2} < 1$, then $c(m_1 \star m_2) < 1$.

It is easily verified that the operator which corresponds to Dempster's rule ($m_1 \star m_2 = \underline{m}_D$) satisfies these basic requirements. Thus if in the considered situation the assumption of the independence of the sources of information is sensible, we should employ Dempster's rule. But if the independence is doubtful, employing this rule can be hazardous, since the conflict is in general pretty high (even if

[5]The bba associated with $bel_{\mathcal{U}}^{\varphi}$ is the function m on $2^{V_{\mathcal{U}}}$ defined by $m(T_{\mathcal{U}}(\varphi)) = 1$ and $m(A) = 0$ if $A \neq T_{\mathcal{U}}(\varphi)$. In particular, $bel_{\mathcal{U}}^{\top}$ is the vacuous belief about $\mathcal{U}$.

the combined beliefs are exactly the same) and this means unnecessary nonmonotonicity.

In order to reduce the unnecessary nonmonotonicity, I suggest to choose the jba which minimizes the conflict (with this choice the monotonicity is assured if it is admissible). If this is not unique, it seems natural to me to choose the least specific one. This is the jba whose respective combination of beliefs maximizes the well established measure of nonspecificity (see for instance Klir and Wierman [7]) among the combinations with respect to the jbas with minimal conflict.

Definition 3 *If bel is a belief about $\mathcal{U}$ with bba m, the measure of nonspecificity of bel is the quantity*

$$N(bel) = \sum_{A \neq \emptyset} m(A) \log_2 |A| .$$

Thus if $c_{\min}^{m_1,m_2} < 1$, I suggest to choose as $m_1 \star m_2$ a jba $\underline{m}$ maximizing $N\big(bel_1 \otimes_{\underline{m}} bel_2\big)$ among the $\underline{m} \in \underline{\mathcal{M}}_{\mathcal{U}}^{m_1,m_2}$ with $c(\underline{m}) = c_{\min}^{m_1,m_2}$ (if $c_{\min}^{m_1,m_2} = 1$, the choice of a jba is useless, since anyway we cannot combine bel_1 and bel_2). From Proposition 3 follows that the task of finding such a $\underline{m}$ is a problem of linear programming.[6]

Proposition 3 *If $m_1, m_2 \in \mathcal{M}_{\mathcal{U}}$, $c_{\min}^{m_1,m_2} < 1$ and $f : \mathbb{N} \longrightarrow \mathbb{R}$ with $f(0) < -|\mathcal{U}|$ and $f(n) = \log_2 n$ for all $n > 0$, then*

$$\arg \max_{\substack{\underline{m} \in \underline{\mathcal{M}}_{\mathcal{U}}^{m_1,m_2} \\ c(\underline{m}) = c_{\min}^{m_1,m_2}}} N\big(bel_1 \otimes_{\underline{m}} bel_2\big) = \arg \max_{\underline{m} \in \underline{\mathcal{M}}_{\mathcal{U}}^{m_1,m_2}} \sum_{A,B \subseteq V_{\mathcal{U}}} \underline{m}(A,B)\, f(|A \cap B|) .$$

Proof. Let $F(\underline{m}) = \sum_{A,B \subseteq V_{\mathcal{U}}} \underline{m}(A,B)\, f(|A \cap B|)$. If $c(\underline{m}) = c_{\min}^{m_1,m_2}$, then

$$F(\underline{m}) = c_{\min}^{m_1,m_2} f(0) + \big(1 - c_{\min}^{m_1,m_2}\big) N\big(bel_1 \otimes_{\underline{m}} bel_2\big) .$$

Therefore it suffices to show that if $\underline{m}$ maximizes $F(\underline{m})$, then $c(\underline{m}) = c_{\min}^{m_1,m_2}$. In the proof of Proposition 2 it is shown that $c(\underline{m}) = c_{\min}^{m_1,m_2}$ is implied by the following property: if $A_1, A_2, B_1, B_2 \subseteq V_{\mathcal{U}}$ with $A_1 \cap B_1 = \emptyset$, $A_1 \cap B_2 \neq \emptyset$, $A_1 \neq A_2$, $\underline{m}(A_1, B_1) > 0$ and $\underline{m}(A_2, B_2) > 0$, then $A_2 \cap B_1 = \emptyset$ and $A_2 \cap B_2 \neq \emptyset$.

Assume that $\underline{m}$ maximizes $F(\underline{m})$ and consider the transformation $\underline{m} \longmapsto \underline{m}'$ defined in the proof of Proposition 2. If the hypothesis of the property stated above holds, we have

$$\begin{aligned} F\big(\underline{m}'\big) &= F(\underline{m}) + \varepsilon\,(f(|A_1 \cap B_2|) + f(|A_2 \cap B_1|) - f(|A_1 \cap B_1|) - f(|A_2 \cap B_2|)) > \\ &> F(\underline{m}) + \varepsilon\,(f(|A_2 \cap B_1|) + |\mathcal{U}| - f(|A_2 \cap B_2|)) . \end{aligned}$$

[6] The proof of Proposition 3 suggests an iteration algorithm for solving this problem: start for instance from $\underline{m}_D$ and recursively apply a transformation of the form $\underline{m} \longmapsto \underline{m}'$ in order to increase the value of the linear functional $\sum \underline{m}(A,B)\, f(|A \cap B|)$. I have not studied the properties of such an algorithm yet.

Therefore $F(\underline{m}') \leq F(\underline{m})$ implies $f(|A_2 \cap B_1|) < 0$ and $f(|A_2 \cap B_2|) \geq 0$; that is, $A_2 \cap B_1 = \emptyset$ and $A_2 \cap B_2 \neq \emptyset$. $\square$

The least specific jba minimizing the conflict is not always unique, thus $m_1 \star m_2$ is not always defined. Consider first the set $\mathcal{S}$ of pairs (m_1, m_2) for which the operator $\star$ is defined: the following properties can be easily verified. In $\mathcal{S}$ the operator $\star$ satisfies the three basic requirements stated above (notice that if $(m_1, m_2) \in \mathcal{S}$, then $(m_1 \circ f, m_2 \circ f) \in \mathcal{S}$ and $(m_1', m_2') \in \mathcal{S}$). If $(m_1, m_2) \in \mathcal{S}$, then $(m_2, m_1) \in \mathcal{S}$ and $m_1 \star m_2\,(A,B) = m_2 \star m_1\,(B,A)$ for all $A, B \subseteq V_{\mathcal{U}}$. If $m \in \mathcal{M}_{\mathcal{U}}$, then $(m,m) \in \mathcal{S}$ and $m \star m\,(A,A) = m(A)$ for all $A \subseteq V_{\mathcal{U}}$. The last two properties imply commutativity and idempotency for the respective combinations of beliefs.

Commutativity is a necessary requirement in symmetrical situations where the two sources of information have the same importance and credibility. In other situations we can prefer that one of the two beliefs has a prominent role in the combination: since these cases can be worked out with other methods (such as discounting), I shall consider commutativity as necessary.

For any pair of bbas $m_1, m_2 \in \mathcal{M}_{\mathcal{U}}$, the least specific jbas minimizing the conflict form a convex polytope (i.e. the bounded intersection of a finite number of closed half-spaces) in $\mathbb{R}^{2^{2|V_{\mathcal{U}}|}}$. Therefore the completion of the definition of the operator $\star$ consists in a rule for assigning to every pair of bbas a point of the respective convex polytope, in such a way that commutativity and the first two basic requirements remain satisfied (the third one being trivially satisfied). Symmetry considerations could lead to the choice of the centre of the polytope (that is, the barycentre with respect to the uniform mass density): this choice fulfills the requirements. Another possibility fulfilling them is for instance the selection of the point of the polytope which minimizes the Euclidean distance from $\underline{m}_D$. I think that the choice of a rule should be based not only on its theoretical properties, but also on considerations about the computational complexity of possible implementations of this rule; since I have not analysed this aspect yet, I leave the question of the completion of the definition of $\star$ open. The contents of the rest of this paper are independent of any particular completion of this definition (such that the above requirements are fulfilled): simply let $\star$ be the obtained operator and let $\odot$ be the respective rule for the combination of beliefs.

Both rules $\oplus$ and $\odot$ satisfy the three basic requirements (to be precise, the corresponding operators satisfy them) and commutativity; $\oplus$ is associative, while $\odot$ is idempotent. Dempster's one is perhaps the only rule of the form $\otimes$ with the four common properties and associativity;[7] anyway, Example 1 shows that associativity and idempotency are two incompatible properties for rules of this form, even if we abandon every other assumption.

[7] The axiomatic derivations of Dempster's rule in Klawonn and Schwecke [5] and Smets [9] do not allow an answer to this question, since both sets of axioms contain a property which is stronger than the ones considered here; while Klawonn and Smets [6] consider a framework which is more restrictive than the one used here.

Example 1 *Let $q \in \mathcal{U}$ and $\frac{1}{2} < \alpha < 1$. Let bel_1 and bel_2 be the minimal beliefs with $bel_1(q) = \alpha$ and $bel_2(\neg q) = \alpha$, respectively. That is, $m_1(Q) = m_2(\overline{Q}) = \alpha$ and $m_1(V_{\mathcal{U}}) = m_2(V_{\mathcal{U}}) = 1 - \alpha$, with $Q = T_{\mathcal{U}}(q)$ and $\overline{Q} = V_{\mathcal{U}} \setminus Q$.*

Then the bba m associated with $bel_1 \otimes bel_2$ satisfies $m(Q) = m(\overline{Q}) = \beta$ and $m(V_{\mathcal{U}}) = 1 - 2\beta$, for a β such that $0 \leq \beta \leq \frac{1}{2}$ (the value of β depends on the choice of a jba).

If we assume idempotency and associativity, we obtain

$$bel_1 \otimes bel_2 = (bel_1 \otimes bel_1) \otimes bel_2 = bel_1 \otimes (bel_1 \otimes bel_2).$$

That is, there is a jba $\underline{m} \in \underline{\mathcal{M}}_{\mathcal{U}}^{m_1,m}$ with

$$m(\overline{Q}) = \frac{\underline{m}(V_{\mathcal{U}}, \overline{Q})}{1 - c(\underline{m})} = \frac{m(\overline{Q}) - \underline{m}(Q, \overline{Q})}{1 - \underline{m}(Q, \overline{Q})}.$$

Therefore $c(\underline{m}) = \underline{m}(Q, \overline{Q}) = 0$, and from Proposition 1 follows that

$$\beta = bel_1 \otimes bel_2(q) \geq bel_1(q) = \alpha,$$

which is a contradiction to $\beta \leq \frac{1}{2} < \alpha$. Thus idempotency and associativity are incompatible (if $|\mathcal{U}| \geq 1$).

In order to combine two beliefs without assuming the independence of the sources, I suggest the rule $\odot$. This can be considered as the most conservative rule of the form $\otimes$: it conserves as much as possible of both beliefs (it has minimal conflict, i.e. maximal monotonicity) without adding anything (it has minimal specificity among the rules with minimal conflict). It is idempotent, thus it cannot be associative. It can be easily verified (for instance by considering epistemic probabilities, defined in Example 3) that associativity is incompatible also with the minimization of the conflict (which is the basic feature of the rule $\odot$).

Idempotency is only a particular case of the following property of the rule $\odot$: if bel_2 is a specialization of bel_1 (i.e. m_2 can be obtained through redistribution of $m_1(A)$ to the non-empty sets $B \subseteq A$, for all $A \subseteq V_U$), then $bel_1 \odot bel_2 = bel_2$. This property is important if strong dependence is possible: if bel_2 is a specialization of bel_1, the information encoded by bel_1 can be part of the information encoded by bel_2, in which case the result of pooling the information is actually bel_2.

Associativity is important because (with commutativity) it implies that the result of the combination of n beliefs is independent of the order in which these beliefs are combined. In a sense, this independence of the order can be obtained also for the rule $\odot$: if we have to combine n beliefs simultaneously, we can consider the set of n-dimensional jbas and extend our rule for the selection of a jba to the n-dimensional case. An interesting problem could be the search for an analogue of Proposition 2 for the n-dimensional case.

Example 2 and Example 3 illustrate the differences between the two rules $\oplus$ and $\odot$ in two simple situations.

Example 2 *Consider the situation of Example 1. Since bel_1 and bel_2 are not compatible, the monotonicity of their combination is not admissible. In fact, for both $\varphi \in \{q, \neg q\}$ we have $\max(bel_1, bel_2)(\varphi) = \alpha > \frac{1}{2}$, while $bel_1 \otimes bel_2(\varphi) = \beta \leq \frac{1}{2}$. Using $\oplus$ we obtain $\beta = \frac{\alpha}{\alpha+1} < \frac{1}{2}$, whereas using $\odot$ we obtain $\beta = \frac{1}{2}$. Thus, unlike the rule $\oplus$, the rule $\odot$ allows only the necessary nonmonotonicity.*

In Example 1 we have seen that no rule of the form $\otimes$ can satisfy both equations $bel_1 \otimes bel_1 = bel_1$ and $(bel_1 \otimes bel_1) \otimes bel_2 = bel_1 \otimes (bel_1 \otimes bel_2)$. Obviously, $\oplus$ satisfies the second one, whereas $\odot$ satisfies the first one. If we want to combine the three beliefs of the second equation in a unique way with the rule $\odot$, we can extend it to the 3-dimensional case. The 3-dimensional jba minimizing the conflict is unique and the respective combination of the three beliefs is the one that we obtain by using the rule $\odot$ in the left-hand side of the equation: $bel_1 \odot bel_2$.

Example 3 *The beliefs bel_1 and bel_2 considered in Example 2 are consonant. In some senses, at the opposite extreme from consonant beliefs we find the epistemic probabilities. A belief about $\mathcal{U}$ with bba m is an epistemic probability if $m(A) = 0$ for all $A \subseteq V_{\mathcal{U}}$ with $|A| \neq 1$. Such a belief is completely defined by the $r = |V_{\mathcal{U}}|$ values $p_1, \ldots, p_r$ that m assigns to the $A \subseteq V_{\mathcal{U}}$ with $|A| = 1$ (it suffices to decide an order for the elements of $V_{\mathcal{U}}$).*

Let bel_1 and bel_2 be two epistemic probabilities defined by $p_1^{(1)}, \ldots, p_r^{(1)}$ and $p_1^{(2)}, \ldots, p_r^{(2)}$, respectively. Then their combination $bel_1 \otimes bel_2$ is still an epistemic probability; let it be defined by $p_1, \ldots, p_r$. The monotonicity is admissible only if $bel_1 = bel_2$, and to assure this monotonicity a rule must be idempotent. Using $\oplus$ we obtain that $p_i = b p_i^{(1)} p_i^{(2)}$ for each $i \in \{1, \ldots, r\}$, where $b \geq 1$ is a normalizing constant. Using $\odot$ we obtain that $p_i = c \min\left\{p_i^{(1)}, p_i^{(2)}\right\} \geq \min\left\{p_i^{(1)}, p_i^{(2)}\right\}$ for each $i \in \{1, \ldots, r\}$, where $c \geq 1$ is a normalizing constant (notice that the inequality is strict unless $bel_1 = bel_2$).

If we want to simultaneously combine n epistemic probabilities defined, respectively, by $p_1^{(j)}, \ldots, p_r^{(j)}$ (for each $j \in \{1, \ldots, n\}$), we can easily extend the rule $\odot$ to the n-dimensional case. The result of the combination is the epistemic probability defined by $p_1, \ldots, p_r$, with $p_i = d \min\left\{p_i^{(1)}, \ldots, p_i^{(n)}\right\}$ for each $i \in \{1, \ldots, r\}$, where $d \geq 1$ is a normalizing constant.

5 A Generalization of Bayes' Theorem

Now I present a situation in which a combination rule minimizing the conflict is especially sensible and in which we can get many results without need to consider the whole combination of beliefs: it suffices to know the value of the conflict between them (which for a combination rule minimizing the conflict can be determined thanks to Proposition 2).

Consider a hypothesis h implying a belief bel about $\mathcal{U}$ (with $h \notin \mathcal{U}$). If we have a belief $bel_{\mathcal{H}}$ about $\mathcal{H} = \{h\}$, we can combine these two beliefs in the following way. We first expand bel to the belief about $\mathcal{U}' = \mathcal{U} \cup \mathcal{H}$ which contains nothing more than the implication $h \Rightarrow bel$: let $(h \Rightarrow bel)$ be the minimal belief about $\mathcal{U}'$ assigning for all $\varphi \in \mathcal{L}_{\mathcal{U}}$ the value $bel(\varphi)$ to the formula $h \rightarrow \varphi$.[8] Then we can combine $(h \Rightarrow bel)$ with the vacuous extension of $bel_{\mathcal{H}}$ to $\mathcal{L}_{\mathcal{U}'}$, obtaining

$$bel_{\mathcal{H}} \upharpoonleft^{\mathcal{U}'} \oplus (h \Rightarrow bel).$$

The use of Dempster's rule is justified in the sense that this is only a formal construction to apply a "metabelief" $bel_{\mathcal{H}}$ about $\mathcal{H}$ to the consequence bel of the hypothesis h (in particular, there can be no conflict). The resulting belief about $\mathcal{U}$ is

$$\left(bel_{\mathcal{H}} \upharpoonleft^{\mathcal{U}'} \oplus (h \Rightarrow bel)\right) \downharpoonleft_{\mathcal{U}} = bel_{\mathcal{H}}(h)\, bel + (1 - bel_{\mathcal{H}}(h))\, bel_{\mathcal{U}}^{\top};$$

that is, the discounting of bel with discount rate $1 - bel_{\mathcal{H}}(h)$. This is sensible, since $pl_{\mathcal{H}}(\neg h) = 1 - bel_{\mathcal{H}}(h)$ measures the amount of our uncertainty about the hypothesis h.

If we get some information in the form of a belief bel' about $\mathcal{U}$, we can combine its vacuous extension to $\mathcal{L}_{\mathcal{U}'}$ with our belief about $\mathcal{U}'$, obtaining in particular a new belief $bel'_{\mathcal{H}}$ about $\mathcal{H}$:

$$bel'_{\mathcal{H}} = \left(\left(bel_{\mathcal{H}} \upharpoonleft^{\mathcal{U}'} \oplus (h \Rightarrow bel)\right) \otimes_{\underline{m}} bel' \upharpoonleft^{\mathcal{U}'}\right) \downharpoonleft_{\mathcal{H}}.$$

Thus in order to get $bel'_{\mathcal{H}}$, we must choose a jba $\underline{m}$. If we reason on the form of the marginal bbas, we can see that $\underline{m}$ is sensible only if it is "naturally" based on a jba $\underline{m}_h$ for the combination of bel and bel'.[9] Then $c(\underline{m}) = bel_{\mathcal{H}}(h)\, c(\underline{m}_h)$, so the combination is possible unless we are sure of the hypothesis and this totally conflicts with the new information (i.e. $bel_{\mathcal{H}}(h) = 1$ and $c(\underline{m}_h) = 1$). The changes in the belief about $\mathcal{H}$ are entirely determined by the conflict $c(\underline{m}_h)$:

$$bel'_{\mathcal{H}}(h) = \frac{bel_{\mathcal{H}}(h) - c(\underline{m})}{1 - c(\underline{m})} \leq bel_{\mathcal{H}}(h) \text{ and}$$

$$bel'_{\mathcal{H}}(\neg h) = \frac{bel_{\mathcal{H}}(\neg h)}{1 - c(\underline{m})} \geq bel_{\mathcal{H}}(\neg h).$$

[8] If m is the bba associated with bel, then the bba associated with $(h \Rightarrow bel)$ is the function m' on $2^{V_{\mathcal{U}'}}$ defined by $m'(T_{\mathcal{U}'}(h \rightarrow \varphi)) = m(T_{\mathcal{U}}(\varphi))$ for all $\varphi \in \mathcal{L}_{\mathcal{U}}$, and $m'(A) = 0$ if $A \notin T_{\mathcal{U}'}(\{h \rightarrow \varphi : \varphi \in \mathcal{L}_{\mathcal{U}}\})$.

[9] If $m_{\mathcal{H}}$ and m' are the bbas associated with $bel_{\mathcal{H}}$ and bel', respectively, then $\underline{m}$ is the jba which satisfies (for all $\varphi, \psi \in \mathcal{L}_{\mathcal{U}}$)

$$\begin{aligned}
\underline{m}(T_{\mathcal{U}'}(h \wedge \varphi), T_{\mathcal{U}'}(\psi)) &= m_{\mathcal{H}}(T_{\mathcal{H}}(h))\, \underline{m}_h(T_{\mathcal{U}}(\varphi), T_{\mathcal{U}}(\psi)), \\
\underline{m}(T_{\mathcal{U}'}(h \rightarrow \varphi), T_{\mathcal{U}'}(\psi)) &= m_{\mathcal{H}}(V_{\mathcal{H}})\, \underline{m}_h(T_{\mathcal{U}}(\varphi), T_{\mathcal{U}}(\psi)) \text{ and} \\
\underline{m}(T_{\mathcal{U}'}(\neg h), T_{\mathcal{U}'}(\psi)) &= m_{\mathcal{H}}(T_{\mathcal{H}}(\neg h))\, m'(T_{\mathcal{U}}(\psi)).
\end{aligned}$$

If $0 < bel_{\mathcal{H}}(h) < 1$, then $bel'_{\mathcal{H}}(h)$ is a strictly decreasing function of $c(\underline{m}_h)$, and in particular we maintain our belief in h only if $c(\underline{m}_h) = 0$. Thus $c(\underline{m}_h)$ (that is, the conflict between the implications of the hypothesis h and the new information) is clearly a measure of disagreement. Therefore it is especially sensible to choose a jba $\underline{m}_h$ minimizing the conflict (and if we are only interested in the new belief about $\mathcal{H}$, then knowing the minimal conflict suffices). With such a choice we obtain in particular that if bel and bel' are compatible, then we maintain our belief in h (this is in general not true if $\underline{m}_h = \underline{m}_D$, even if $bel = bel'$).

Consider now the general case with n hypotheses $h_1, \ldots, h_n$ implying, respectively, the beliefs $bel_1, \ldots, bel_n$ about $\mathcal{U}$ (with $h_1, \ldots, h_n \notin \mathcal{U}$). Given an "a priori" belief $bel_{\mathcal{H}}$ about $\mathcal{H} = \{h_1, \ldots, h_n\}$ and an "observation" belief bel' about $\mathcal{U}$, we can combine these beliefs to obtain an "a posteriori" belief $bel'_{\mathcal{H}}$ about $\mathcal{H}$:

$$bel'_{\mathcal{H}} = \left(\left(bel_{\mathcal{H}} \upharpoonright^{\mathcal{U}'} \oplus \bigotimes_{i=1}^{n} (h_i \Rightarrow bel_i) \upharpoonright^{\mathcal{U}'} \right) \otimes_{\underline{m}} bel' \upharpoonright^{\mathcal{U}'} \right) \downarrow_{\mathcal{H}} .$$

As before, $\mathcal{U}' = \mathcal{U} \cup \mathcal{H}$ and the use of Dempster's rule in the first combination can be justified as a formal construction. The new element is $\bigotimes_{i=1}^{n} (h_i \Rightarrow bel_i) \upharpoonright^{\mathcal{U}'}$, which is any combination of the n beliefs $(h_i \Rightarrow bel_i) \upharpoonright^{\mathcal{U}'}$ using the general rule $\otimes$ (we can obtain it by $n-1$ applications of the binary rule or with a n-dimensional jba). This allows the hypotheses to be dependent (for instance if two hypotheses differ only by a detail and the two implied beliefs are almost the same), and it is important to notice that anyway there can be no conflict among the $n+1$ beliefs $bel_{\mathcal{H}} \upharpoonright^{\mathcal{U}'}$ and $(h_i \Rightarrow bel_i) \upharpoonright^{\mathcal{U}'}$.

This way to update a belief about $\mathcal{H}$ is a broad generalization of Bayes' theorem for epistemic probabilities and of Smets' generalized Bayesian theorem (gBt) for normalized beliefs (see for instance Smets [11]). The construction of $\bigotimes_{i=1}^{n} (h_i \Rightarrow bel_i) \upharpoonright^{\mathcal{U}'}$ allows a lot of freedom, which of course can be limited by some additional assumptions. Before introducing two such assumptions, I consider a simple special case.

Let $bel_{\mathcal{H}}$ be a belief about $\mathcal{H}$ satisfying

$$\sum_{i=0}^{n} bel_{\mathcal{H}}(\varphi_i) = 1, \text{ where}$$
$$\varphi_0 = \neg h_1 \wedge \ldots \wedge \neg h_n \text{ and}$$
$$\varphi_i = \neg h_1 \wedge \ldots \wedge \neg h_{i-1} \wedge h_i \wedge \neg h_{i+1} \ldots \wedge \neg h_n \text{ if } i \in \{1, \ldots, n\};$$

that is, $bel_{\mathcal{H}}$ is an epistemic probability on $\varphi_0, \ldots, \varphi_n$. Then

$$bel_{\mathcal{H}} \upharpoonright^{\mathcal{U}'} \oplus \bigotimes_{i=1}^{n} (h_i \Rightarrow bel_i) \upharpoonright^{\mathcal{U}'}$$

is independent of the choice of $\bigotimes_{i=1}^{n}(h_i \Rightarrow bel_i)\upharpoonright^{\mathcal{U}'}$, and its restriction to $\mathcal{L}_{\mathcal{U}}$ is

$$\sum_{i=1}^{n} bel_{\mathcal{H}}(h_i)\, bel_i + bel_{\mathcal{H}}(\varphi_0)\, bel_{\mathcal{U}}^{\top}.$$

This shows that φ_0 can be considered as an additional hypothesis h_0 implying the vacuous belief $bel_0 = bel_{\mathcal{U}}^{\top}$, and $bel_{\mathcal{H}}$ can be seen as an epistemic probability on the mutually exclusive and exhaustive hypotheses $h_0,\ldots,h_n$. As before, in order to get $bel'_{\mathcal{H}}$, we must choose a jba $\underline{m}$. And as before, if we reason on the form of the marginal bbas, we can see that $\underline{m}$ is sensible only if it is "naturally" based on the jbas $\underline{m}_i$ of the combinations of bel_i and bel' (for each $i \in \{0,\ldots,n\}$).[10] Then $c(\underline{m}) = \sum_{i=1}^{n} bel_{\mathcal{H}}(h_i)\, c(\underline{m}_i)$ (notice that $c(\underline{m}_0) = 0$), so the combination is possible unless we are sure that the truth is among some hypotheses and these totally conflict with the new information. The belief about $\mathcal{H}$ remains an epistemic probability on $\varphi_0,\ldots,\varphi_n$, and as before, the changes are entirely determined by the conflicts $c(\underline{m}_1),\ldots,c(\underline{m}_n)$:

$$bel'_{\mathcal{H}}(h_i) = \frac{1-c(\underline{m}_i)}{1-c(\underline{m})} bel_{\mathcal{H}}(h_i) \text{ for each } i \in \{0,\ldots,n\}.$$

Thus the belief in a hypothesis h_i increases if and only if the respective conflict $c(\underline{m}_i)$ is less than $c(\underline{m})$, which is a weighted average of the conflicts of the $n+1$ hypotheses (h_0 included). Therefore the conflicts $c(\underline{m}_1),\ldots,c(\underline{m}_n)$ measure the disagreement between the respective hypotheses and the new information, and thus it is especially sensible to choose jbas $\underline{m}_1,\ldots,\underline{m}_n$ minimizing the conflict. With such a choice we obtain in particular that if bel_i and bel' are compatible, then the belief in h_i does not decrease (and it increases if $c(\underline{m}) > 0$); this is in general not true if $\underline{m}_i = \underline{m}_D$, even if $bel_i = bel'$.

I now consider the two announced assumptions which limit the freedom in the construction of $\bigotimes_{i=1}^{n}(h_i \Rightarrow bel_i)\upharpoonright^{\mathcal{U}'}$. The first one is that the hypotheses $h_1,\ldots,h_n$ are mutually exclusive (i.e. $bel_{\mathcal{H}}(\varphi_0 \vee \ldots \vee \varphi_n) = 1$), but not necessarily exhaustive (which would mean $bel_{\mathcal{H}}(\varphi_1 \vee \ldots \vee \varphi_n) = 1$). The second one is that the beliefs $bel_1,\ldots,bel_n$ are issued from independent sources of information (the sources can be identified with the respective hypotheses $h_1,\ldots,h_n$). Since the hypotheses are mutually exclusive, this simply means that the belief about $\mathcal{U}$ implied

[10] $\underline{m}$ is the jba which satisfies (for all $\varphi,\psi \in \mathcal{L}_{\mathcal{U}}$ and $i \in \{0,\ldots,n\}$)

$$\underline{m}(T_{\mathcal{U}'}(\varphi_i \wedge \varphi), T_{\mathcal{U}'}(\psi)) = bel_{\mathcal{H}}(\varphi_i)\, \underline{m}_i(T_{\mathcal{U}}(\varphi), T_{\mathcal{U}}(\psi)).$$

by a disjunction of hypotheses is the disjunctive combination of the respective beliefs (the disjunctive rule of combination is defined for instance in Smets [11]). With these two additional assumptions, we obtain

$$bel'_{\mathcal{H}} = \left(\left(bel_{\mathcal{H}} \upharpoonleft^{\mathcal{U}'} \oplus \bigoplus_{i=1}^{n} (h_i \Rightarrow bel_i) \upharpoonleft^{\mathcal{U}'} \right) \otimes_{\underline{m}} bel' \upharpoonleft^{\mathcal{U}'} \right) \downarrow_{\mathcal{H}} .$$

This is a generalization of Smets' gBt for normalized beliefs (which corresponds to the special case with $\underline{m} = \underline{m}_D$), and thus also of Bayes' theorem for epistemic probabilities. If bel' has the form $bel^{\varphi}_{\mathcal{U}}$ (in the literature the gBt is usually restricted to this case), the jba $\underline{m}$ is unique and the updated belief $bel'_{\mathcal{H}}$ is the one that we would obtain by applying the gBt to the $n+1$ hypotheses $h_0, \ldots, h_n$ (with $bel_0 = bel^{\top}_{\mathcal{U}}$). But if bel' has not the form $bel^{\varphi}_{\mathcal{U}}$, we must choose a jba $\underline{m}$; and as before, $\underline{m}$ can be sensible only if it is "naturally" based on the jbas of the combinations of the new information bel' with the beliefs implied by the hypotheses or by any disjunction of hypotheses. Since also in this more general case the conflicts measure the disagreement between the respective hypotheses (or disjunctions of hypotheses) and the new information, it is especially sensible to choose jbas minimizing the conflict. With such a choice we obtain in particular that if the beliefs implied by some hypotheses are compatible with the new information, then the values of the belief in these hypotheses and in their disjunctions do not decrease (and they increase if $c(\underline{m}) > 0$). If instead we use Dempster's rule (that is, we use the gBt), we can get very bad results, since the conflict between the new information bel' and a hypothesis h implying the belief bel can be very high, even if $bel' = bel$ (i.e. the prevision of h is perfect). In fact, if a hypothesis is correct, can we assume that the belief which is a theoretical consequence of the hypothesis and the belief which is a practical consequence of the correctness of the hypothesis are independent?

6 Conclusion

In this paper a rule has been proposed to combine two belief functions issued from sources of information whose independence is doubtful. This rule increases the monotonicity of the reasoning, assuring in particular complete monotonicity if this is admissible. The proposed combination rule is commutative and idempotent. It is not associative, but it can be easily extended to a rule for the simultaneous combination of any number of belief functions.

The proposed combination rule leads to sensible results in a generalization of Bayes' theorem for epistemic probabilities and of Smets' generalized Bayesian theorem. This generalization allows the new information to be any belief function: in this situation the use of Dempster's rule (that is, the independence assumption) leads to questionable results.

References

[1] B. Ben Yaghlane, P. Smets and K. Mellouli. Belief Function Independence: I. The Marginal Case. *International Journal of Approximate Reasoning*, 29:47–70, 2002.

[2] A. P. Dempster. Upper and Lower Probabilities Induced by a Multivalued Mapping. *The Annals of Mathematical Statistics*, 38:325–339, 1967.

[3] D. Dubois and H. Prade. On the Unicity of Dempster Rule of Combination. *International Journal of Intelligent Systems*, 1:133–142, 1986.

[4] R. L. Epstein. *Propositional Logics*. The Semantic Foundations of Logic, 2nd edition, Wadsworth, 2001.

[5] F. Klawonn and E. Schwecke. On the Axiomatic Justification of Dempster's Rule of Combination. *International Journal of Intelligent Systems*, 7:469–478, 1992.

[6] F. Klawonn and P. Smets. The Dynamic of Belief in the Transferable Belief Model and Specialization-Generalization Matrices. In Dubois, Wellman, D'Ambrosio and Smets, editors, *Proceedings of the 8th Conference on Uncertainty in Artificial Intelligence*, 130–137, Morgan Kaufmann, 1992.

[7] G. J. Klir and M. J. Wierman. *Uncertainty-Based Information: Elements of Generalized Information Theory*. Studies in Fuzziness and Soft Computing, 15, 2nd edition, Physica-Verlag, 1999.

[8] G. Shafer. *A Mathematical Theory of Evidence*. Princeton University Press, 1976.

[9] P. Smets. The Combination of Evidence in the Transferable Belief Model. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 12:447–458, 1990.

[10] P. Smets. The Concept of Distinct Evidence. In Bouchon-Meunier, Valverde and Yager, editors, *Proceedings of the 4th Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems*, 789–794, Lecture Notes in Computer Science, 682, Springer, 1993.

[11] P. Smets. Belief Functions: The Disjunctive Rule of Combination and the Generalized Bayesian Theorem. *International Journal of Approximate Reasoning*, 9:1–35, 1993.

[12] R. R. Yager. Nonmonotonic Reasoning with Belief Structures. In Yager, Kacprzyk and Fedrizzi, editors, *Advances in the Dempster-Shafer Theory of Evidence*, Chapter 24, 533–554, Wiley, 1994.

Marco E.G.V. Cattaneo is with the Seminar for Statistics, Department of Mathematics, ETH Zürich, Switzerland. E-mail: cattaneo@stat.math.ethz.ch