Independence with Respect to Upper and Lower Conditional Probabilities Assigned by Hausdorff Outer and Inner Measures

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Abstract

Upper and lower conditional probabilities assigned by Hausdorff outer and inner measures are given; they are *natural extensions* to the class of all subsets of Ω=[0,1] of finitely additive conditional probabilities, in the sense of Dubins, assigned by a class of Hausdorff measures. A *weak disintegration property* is introduced when conditional probability is defined by a class of Hausdorff dimensional measures. Moreover the definition of *s-independence* and *s-irrelevance* are given to assure that logical indepedence is a necessary condition of independence. The interpretation of *commensurable* events in the sense of de Finetti as sets with finite and positive Hausdorff measure and with the same Hausdorff dimension is proposed.

Keywords

upper and lower conditional probabilities, Hausdorff measures, disintegration property, independence

1 Introduction

The necessity to introduce a new tool to assess conditional probabilities is due to some problems related to the axiomatic definition of regular conditional probability (or regular conditional distribution) Q(A,ω) on a σ-field **F** given a sub σ-field **G**. A regular conditional probability can not exist [7]; moreover even if it exists, if **F** is a σ-field countably generated and **G** is sub σ-field of **F** not countably generated, than there exists no regular, *proper* conditional probability Q(A,ω) on **F** given **G**, that is Q(H,ω) =1 for $\omega \in H \in$**G** ([2], [3]). In a recent paper of Seidenfeld, Schervish and Kadane [16] *improper regular conditional distributions* are studied. The authors established that when regular conditional probability exists and the sub σ-field **G** is countably generated almost surely it is proper, but when the sub σ-field **G** is not countable generated the regular conditional probability can be *maximally improper*, that is Q(H,ω) =0 for $\omega \in H \in$**G**, almost surely. Alternative probabilistic approaches that always assure the existence of a proper conditional probability are those proposed by de Finetti [5, 6], Dubins [9] and Walley [17]. In [8] finitely additive conditional probabilities in the sense of Dubins are given by a class of Hausdorff dimensional measures. Their *natural extensions* are given in Section 2 of this paper by outer and inner Hausdorff measures. In particular the case where the σ-field of the conditioning events is not countable generated is analysed. In fact we consider **G** equal to the σ-field of countable or co-countable sets, to the tail σ-field and equal to the σ-field of symmetric events. A problem related to the theory of finitely additive conditional probability is that it does not always satisfy the disintegration property. In section 3 we analyse the meaning of the disintegration property when conditional probability is assigned by a class of Hausdorff dimensional measures. In particular a *weak disintegration property* is introduced and it is proved that this property is verified by conditional probability assigned by a class of Hausdorff measures. There is another reason to investigate coherent conditional probabilities: it is that, some paradoxical situations about stochastic independence, can be solved if a stronger definition of independence, tested with respect to upper and lower conditional probabilities assigned by outer and inner Hausdorff measure, is given. To this aim in section 4 we introduce the definitions of *s-independence* and *s-irrelevance* that are based on the fact that epistemic independence and irrelevance, introduce by Walley, must be tested for events A and B such that the intersection A∩B and the events A and B have the same Hausdorff dimension. With this further condition we prove that s-independence implies logical independence. The results proposed in this paper are based on the idea that *commensurable* events in the sense of de Finetti [4], are subsets of Ω with the same Hausdorff dimension when conditional probability is assigned by a class of Hausdorff measures. At the end of this paper we put in evidence the possibility to use conditional probabilities, assigned by Hausdorff dimensional measures, to deal uncertainty in complex natural phenomena and to give hazard assessments. In fact in different fields of science (geology, biology, architecture) many data sets are fractal sets, i.e. are sets with non-integer Hausdorff dimension. So conditional probabilities, assigned by a Hausdorff dimensional measures, can be used as tool to make inference given fractal sets of data.

2 Upper and Lower Conditional Probabilities Assigned by Hausdorff Outer and Inner Measures

In Walley [17] (Chap. 6) coherent conditional probabilities are considered as a special case of coherent conditional previsions, that are characterized in the case where conditioning events form a partition **B** of Ω.The real number $\overline{P}(X|B)$ are specified for B in **B** and all gambles X in some domain H(B). Conditional previsions $\overline{P}(X|B)$, defined for B in **B** and all gambles X in H(B), are s*eparately* *coherent* when for every conditioning event B, $\overline{P}(\cdot|B)$ is a coherent upper prevision on the domain H(B) and $\overline{P}(B|B) = 1$.

When the domain H(B) is a class of events, that can be regarded as a class of 0-1 valued gambles, $\overline{P}(X|B)$ is a coherent upper conditional probability. In particular when $P(\cdot|B)$ is a countably additive probability defined on a σ-field, its *natural extensions* to the class of all subsets of Ω, called coherent upper and lower probabilities are the outer and inner measures generated by it (see Theorem 3.1.5 of [17]).

In the standard theory, conditional previsions P(X|**G**) are defined with respect to a σ-field of events **G**, rather then a partition **B.** The two approches are closely related when **G** is the σ-field made up of all unions of sets in **B.**

In this section coherent upper and lower conditional probabilities are given by the inner and outer measures generated by the Hausdorff dimensional measures. They are *natural extensions* to the class of all subsets of Ω=[0,1] of finitely additive conditional probabilities, in the sense of Dubins [9] assigned by a class of Hausdorff measures.

Let **F** and **G** be two fields of subsets of Ω, with **G**⊆**F**, P* is a *finitely additive conditional probability* [9] on (**F**,**G**) if it is a real function defined on $\mathbf{F}\times\mathbf{G}^0$, where $\mathbf{G}^0$= **G**-{∅} such that the following conditions hold:

I) given any $H\in G^0$ and $A_1,...,A_n \in \mathbf{F}$ with $A_i \cap A_j = \emptyset$ for i≠j, the function P*(·|H) defined on **F** is such that

$$P*(A|H) \geq 0, P*(\bigcup_{k=1}^{n} A_k|H) = \sum_{k=1}^{n} P^*(A_k|H), P*(|H) = 1$$

II) P*(H|H)=1 if $H\in\mathbf{F}\cap\mathbf{G}^0$

III) given E∈**F**, H∈**F**, EH∈**F** with $A\in\mathbf{G}^0$ and $EA\in\mathbf{G}^0$ then

P*(EH|A)=P*(E|A)P*(H|EA).

From conditions I) and II) we have

II') P*(A|H)=1 if A∈**F**, $H\in\mathbf{G}^0$and H⊂A.

These conditional probabilities are coherent in the sense of de Finetti, since conditions I), II), III) are sufficient [14] for the coherence of P* on $\mathbf{C}=\mathbf{F}\times\mathbf{G}^0$ when **F** and **G** are fields of subsets of Ω with **G**⊆ **F** or when **G** is an additive subclass of **F**; otherwise if **F** and **G** are two arbitrary families of subsets of Ω, such that Ω ∈**F** the previous conditions are necessary for the coherence [11, 14], but not sufficient.

Now we recall some definitions about Hausdorff dimensional outer measures that we use as tool to give upper conditional probabilities (for more details about Hausdorff measures see for example [10]).

Let (Ω,d) be the Euclidean metric space with Ω=[0,1]. The diameter of a nonempty set U of Ω is defined as |U|=sup{|x-y|: x,y∈U} and if a subset A of Ω is such that $A\subset \bigcup_i U_i$ and $0< |U_i| < \delta$ for each *i*, the class $\{U_i\}$ is called a δ-cover of

A. Let s be a non-negative number. For $\delta > 0$ we define $h^s(A) = \inf \sum_{i=1}^{\infty} |U_i|^s$, where the infimum is over all (countable) δ-covers $\{U_i\}$. The Hausdorff s-dimensional outer measure of A, denoted by $h^s(A)$, is defined as $h^s(A) = \lim_{\delta \to 0} h^s_\delta(A)$. This limit exists, but may be infinite, since $h^s_\delta(A)$ increases as δ decreases.

The Hausdorff dimension of a set A, $\dim_H(A)$, is defined as the unique value, such that $h^s(A) = \infty$ if $0 \leq s < \dim_H A$ and $h^s(A) = 0$ if $\dim_H A < s < \infty$. We can observe that if $0 < h_s(A) < \infty$ then $\dim_H(A) = s$, but the converse is not true. We assume the Hausdorff dimension of the empty set equal to -1. So no event has Hausdorff dimension equal to the empty set.

Remark: It is important to note the link between the Hausdorff dimension of an event and the Hausdorff dimension of its complement. In fact, denoted by $\dim_H(A)$ the Hausdorff dimension of A we have [10] that

$$dim_H(A \cup B) = max\{dim_H(A), dim_H(B)\};$$

in particular if $A = B^c$ we obtain that $1 = \dim_H(\Omega) = \max\{\dim_H(B), \dim_H(B^c)\}$; so if $\dim_H(B) < \dim_H(B^c)$ then $\dim(B^c) = 1$.

Upper conditional probabilities are given by outer Hausdorff dimensional measures, firstly in the case where conditioning events have finite and positive Hausdorff outer measure.

Theorem 1 *Let Ω=[0,1] and let **F** be the σ-field of all subsets of [0,1] and let **G** be an additive subclass of **F** of sets such that for every H in **G** we have $0 < h^s(H) < \infty$, where s is the Hausdorff dimension of H and h_s is the Hausdorff s-dimensional outer measure. Then for each H in **G** the real function $P(\cdot|H)$ defined on **F**, such that*

$$\overline{P}(A|H) = \frac{h^s(AH)}{h^s(H)}$$

verifies the following properties:

a) $0 \leq \overline{P}(A|H) \leq 1$;

b) $\overline{P}(A \cup B|H) \leq \overline{P}(A|H) + \overline{P}(B|H)$ and $\overline{P}(A \cup B|H) = \overline{P}(A|H) + \overline{P}(B|H)$ whenever A and B are positively separated, that is $d(A,B) = \inf\{d(x,y): x \in A, y \in B\} > 0$;

c) for each $H \in G$ $\overline{P}(\cdot|H)$ is a coherent upper probability.

Proof. For each H belonging to **G** we have, for the monotony of the Hausdorff outer measures, that

$$0 \leq \overline{P}(A|H) = \frac{h^s(AH)}{h^s(H)} \leq \frac{h^s(H)}{h^s(H)} = 1;$$

Moreover, since h^s is an outer measure for every s then it is subadditive. For every s the Hausdorff outer measure h^s is a *metric outer measure* that is $h^s(A \cup B) = h^s(A) + h^s(B)$ whenever A and B are positively separated.

Property c) follows from Theorem 3.1.5. of [17]. □

In the general case, when conditioning events can have infinite or zero Hausdorff measure, conditional probability is defined by a 0-1 valued finitely additive (but not countable additive) probability measure m; this assures condition III) of a finitely conditional probability in the sense of Dubins, is verified.

Theorem 2 *Let Ω=[0,1], let **F** be the σ-field of all subsets of [0,1] and let **G** be an additive sub-class of **F**. Let us denoted by h^s the Hausdorff s-dimensional outer measure, by s the Hausdorff dimension of H and by t Hausdorff dimension of AH; let m be a 0-1 valued finitely additive (but not countable additive) probability measure. Then the function $\overline{P}$ defined on **C**=**F**×$\mathbf{G}^0$ such that*

$$\overline{P}(A|H) = \begin{cases} \frac{h^s(AH)}{h^s(H)} & if \quad 0 < h^s(H) < \infty \\ m(AH) & if \quad h^s(H) = 0, \infty \end{cases}$$

is an upper conditional probability.

Proof. Firstly we prove that the restriction of $\overline{P}$ to the Cartesian product of **B**×$\mathbf{G}^0$, where **B** is the Borel σ-field of [0,1] is a coherent conditional probability. The restriction of the Hausdorff s-dimensional outer measure to the σ-field of the borelian sets of [0,1] is a measure for every s so, by definition, we have, that $\overline{P}$ (·|H) verifies condition I) and II).

To prove condition III), that is $\overline{P}$(EH|A)=$\overline{P}$(E|A)$\overline{P}$(H|EA), for E∈**B**, H∈**B** EH∈**B** with A∈$\mathbf{G}^0$ and EA∈$\mathbf{G}^0$, we distinguish the following cases:

a) conditioning events A and EA have positive and finite Hausdorff measures, then condition III) can be written as

$$\frac{\mathrm{h}^s(\mathrm{EAH})}{\mathrm{h}^s(\mathrm{A})} = \frac{\mathrm{h}^s(\mathrm{EA})}{\mathrm{h}^s(\mathrm{A})} \cdot \frac{\mathrm{h}^t(\mathrm{EAH})}{\mathrm{h}^t(\mathrm{EA})} \tag{1}$$

Two cases are possible: i) $s = t$ or ii) $s > t$.

If i) holds than (1) is obviously satisfied. If ii) holds than h^s(EA)=0 and also, by the monotony of h^s, h^s(EAH)=0; so equation (1) is satisfied.

b) conditioning events A and EA have both infinite or zero Hausdorff measures then condition III) becomes m(EAH)=m(EAH)m(EA) and it is always satisfied because m is monotone;

c) conditioning event A has infinite Hausdorff measure and conditioning event EA has positive and finite Hausdorff measure then from the definition of m it follows that condition III) becomes 0=0 , and it is obviously satisfied.

Then from Theorem 3.1.5 of [16] we have that if 0<h^s(H)< ∞ then $\overline{P}$ is the natural extension to **C**=**F**×$\mathbf{G}^0$ moreover if h^s(H)=0 or ∞ then m can be extended to **C**=**F**×$\mathbf{G}^0$ since m is finitely additive, but not countable additive. □

The upper conditional probability defined in the previous Theorem 2 can be used to assess conditional upper probabilities when the class of conditioning events is not a countably generated σ-field. In particular if **G** is equal to the σ-field of countable or co-countable sets, to the tail σ-field or to the σ-field of symmetric events. In all these cases conditioning events have Lebesgue measure equal to one or zero. So upper conditional probability can be defined as in Theorem 2.

Example 1 *Let (Ω,**F**,P) be a probability space where Ω=[0,1], **F** is the σ-field of Borel of Ω and P is the Lebesgue measure on **F**. Let **G** be the sub σ-field of **F** of sets that are either countable or co-countable. Since the probability of the events of the σ-field **G** is either 0 or 1, we have that the probability of A given **G** is equal to P(A), with probability 1, if conditional probability is defined by the Radon-Nikodym derivative. That is*

$$P[A||\mathbf{G}]_\omega = P(A)$$

except on a P zero subset of [0,1].

Given A=[a,b] with $0 < a < b < 1$ let P^* be the real function defined on **C**=**F**×$\mathbf{G}^0$ such that the restriction P^*_r to **E**=$\{(A,\{\omega\}):\omega \in [0,1]\}$ is equal, with probability 1, to the Radon-Nikodym derivative P[A||**G**] $_\omega$. We have that P* is not coherent on **C**, since it does not satisfy the property that P*(A,{ω}) is equal to 1 or 0 according to whether ω belongs to A or not.

A finitely additive conditional probability on **C**=**F**×$\mathbf{G}^0$ can be defined by

$$\overline{P}(A|H) = \begin{cases} \frac{h^1(AH)}{h^1(H)} & H \quad \text{co-countable} \\ \frac{h^0(AH)}{h^0(H)} & H \quad \text{finite} \\ m(AH) & H \quad \text{countable} \end{cases}$$

where m is a 0-1 valued finitely additive (but not countably additive) probability measure.

The function $\overline{P}$ is a coherent conditional probability since it verifies the axioms of a finitely additive probability in the sense of Dubins as proved in Theorem 2.

The lower conditional probability $\underline{P}$(A|H) can be define as in the previous theorems if h^s denotes the Hausdorff s-dimensional inner measure.

3 The Disintegration Property

In this section we analyse the meaning of the disintegration property when conditional probability is assigned by a class of Hausdorff dimensional measures. In particular a *weak disintegration property* is introduced. If conditional probability is defined by the Radon-Nikodym derivative $P[A||G]_\omega$, it verifies the disintegration property, that is the functional equation $P(A\cap H)=\int_H P[A||G]_\omega dP$ with H∈**G**.

This property is not always satisfied in the theory of finitely additive probability of Dubins. In fact with a finitely additive probability P it is not assured that P(A)=

$\int_{\Omega}$P[A||G]$_\omega$dP for A in **F**. In the paper of Schervish, Seidenfeld and Kadane [15] has been shown that each finitely but not countably additive probability P will fail to be disintegrable on some denumerable partition of Ω.

Let Ω=[0,1], let **F** be the σ-field of the Borel subsets of [0,1], **G** a sub σ-field of **F** and let P be equal to h^1, that is the Lebesgue measure. We recall that since the class of subsets of Ω measurable with respect to h^s, for every s, is the class of Borel subsets of [0,1], than each h^s is a measure (σ-additive) on **F**. We denote by P* the restriction to **F**×$\mathbf{G}^0$, of the upper conditional probability assigned in Theorem 2. For each H in $\mathbf{G}^0$P$^*(A|H)$ is a function on H.

The starting point is that when the conditioning event H has Hausdorff dimension s less then 1, the equation P(A∩H)=$\int_H$P*(A|H)dP is obviously verified since dim(A∩H)≤dim(H)<1 then P(A∩H)=0=P(H) and $\int_H$P*(A|H)dP=0. So it can be interesting to investigate if an analogous equation holds with respect to the measure h^s. We observe that, if h^s(H)=∞ then the functions P*(A|H), defined in the previous section, are not integrable since no constant different from zero is integrable on H with respect to h^s; so we introduce the following definition

Definition 1. Let Ω=[0,1], let **F** be the σ-field of the Borel subsets of [0,1] and let P be equal to h^1, that is the Lebesgue measure. Let **G** be a sub-σ−field of **F**. Denoted by h^s the Hausdorff s-dimensional measure where s is the Hausdorff dimension of H. A coherent conditional probability P* verifies the *weak disintegration property* if the following functional equation h^s(A∩H)=$\int_H$P*(A|H)dh^s is verified for every H in $\mathbf{G}^0$ with h^s(H)< ∞.

Remark: If dim(A∩H)<dim(H)=s and h^s(H)< ∞ then the equation

$$h^s(A\cap H)=\int_H P*(A|H)dh^s$$

is satisfied since both members are equal to zero. So to verify that a given coherent conditional probability satisfied the weak disintegration property we have to prove that the equation is verified for every pair of event A, H with dim(AH)=dim(H).

Theorem 3 *Let Ω=[0,1], let **F** be the σ-field of the Borel subsets of [0,1] and let P be equal to h^1, that is the Lebesgue measure. Let **G** be a sub-σ-field of **F**. Having fixed A in **F**, let us denoted by h^s the Hausdorff s-dimensional measure, by s the Hausdorff dimension of H; let m be a 0-1 valued finitely additive (but not countable additive) probability measure. The coherent conditional probability P* defined on **C**=**F**×$\mathbf{G}^0$ such that*

$$P^*(A|H)=\begin{cases}\frac{h^s(AH)}{h^s(H)} & if \quad 0<h^s(H)<\infty\\ m(AH) & if \quad h^s(H)=0,\infty\end{cases}$$

verifies the weak disintegration property.

Proof. We have to prove that the equation

$$h^s(A \cap H) = \int_H P^*(A|H)dh^s \tag{2}$$

is verified for every H in $\mathbf{G}^0$ with $h^s(H) < \infty$.

Firstly we suppose $h^s(H)$ positive and finite; for each A and H, the function P*(A|H) is nonnegative and less or equal to 1, so it is integrable with respect to h^s; then we observe that the equation (2) is always satisfied since

$$\int_H P^*(A|H)dh^s = \int_H \frac{h^s(A \cap H)}{h^s(H)} dh^s = h^s(A \cap H).$$

Moreover if $h^s(H)$ is equal to zero, then equation (2) vanishes to 0=0. □

4 Independence

In this section we introduce a new definition of independence for events, called *s-independence*, based on the fact that the relative events and their intersection must have the same Hausdorff dimension. This notion does not require any assumption of positivity for the probability of the conditioning event. This is one of the difference with the concepts of *confirmational irrelevance* and *strong confirmational irrelevance,* proposed by Levi [12].

We prove that s-independence between events implies their logical independence when both events have Hausdorff dimension less than 1. Moreover also when the events have Hausdorff dimension equal to 1 and positive and finite Lebesgue outer measure then logical dependence is a necessary condition for the s-independence. Firstly we analyse the concept of *epistemically independence* for events proposed by Walley [17] with respect to conditional upper and lower probabilities defined by Hausdorff dimensional outer and inner mesures. The concept of epistemic independence is based on the notion of *irrelevence*; given two events A and B, we say that B is *irrelevant* to A when $\underline{P}(A|B)=\underline{P}(A|B^c)=\underline{P}(A)$ and $\overline{P}(A|B)=\overline{P}(A|B^c)=\overline{P}(A)$.

A and B are *epistemic independent* when B is irrelevant to A and A is irrelevant to B. As a consequence of this definition we can obtain the factorisation property P(A∩B)=P(A)P(B) that constitutes the standard definition of independence for events. Let Ω=[0,1] and let $\overline{P}$ and $\underline{P}$ be the upper and lower conditional probabilities defined by the outer and inner Hausdorff measures. The unconditional upper and lower probabilities can be obtained from the conditional ones by the equalities $\overline{P}(A)=\overline{P}(A|\Omega)$= and $\underline{P}(A)=\underline{P}(A|\Omega)$.

When the events A and B or their complements have not upper probability equal to zero, epistemic independence implies *logical independence*, (i.e. each of four sets A∩B, A∩B^c, A^c∩B, A^c∩B^c are non-empty). Otherwise logically dependent events can be epistemically independent.

Example 2 *Let Ω=[0,1], let* ***F*** *be the σ-field of all subsets of [0,1] and let* ***G*** *be the additive sub-class of* ***F*** *of sets that are finite and co-finite. Let A and B two finite subsets of [0,1] such that A∩B=∅. If conditional probability is defined as in Theorem 2 we have that*

$$\underline{P}(A|B) = \overline{P}(A|B) = \frac{h^0(AB)}{h^0(B)} = 0$$
$$\underline{P}(A|B^c) = \overline{P}(A|B^c) = \frac{h^1(AB^c)}{h^1(B^c)} = 0$$
$$\underline{P}(A) = \overline{P}(A) = \overline{P}(A|\Omega) = \frac{h^1(A)}{h^1(\Omega)} = 0$$

So A and B are logical dependent but epistemically independent.

The previous example puts in evidence the necessity to introduce the following definition.

Definition 2. Let Ω=[0,1], let **F** be the σ-field of all subsets of [0,1] and let **G**=**F**. Denoted by $\overline{P}$ and $\underline{P}$ be the upper and lower conditional probabilities defined by the outer and inner Hausdorff measures and given A and B in $\mathbf{G}^0$, then they are *s-independent* if the following conditions hold:

1) $\dim_H$(AB)=$\dim_H$(B)=$\dim_H$(A)

2) $\underline{P}$(A|B)=$\underline{P}$(A|B^c)=$\underline{P}$(A) and $\overline{P}$(A|B)=$\overline{P}$(A|B^c)=$\overline{P}$(A)

3) $\underline{P}$(B|A)=$\underline{P}$(A|A^c)=$\underline{P}$(B) and $\overline{P}$(B|A)=$\overline{P}$(B|A^c)=$\overline{P}$(B)

Remark: Two disjoint events A and B are s-dependent since the Hausdorff dimension of the empty set can not be equal to that one of any other set so condition 1 is never satisfied. In particular the events A and B of Example 1, that are logical dependent but epistemically independent, are not s-independent.

We prove that logical independence between two events A and B is a necessary condition for s-independence when $\dim_H$(A) and $\dim_H$(B) are both less then 1.

Theorem 4 *Let Ω=[0,1], let* ***F*** *be the σ-field of all subsets of [0,1], let* ***G***=***F*** *and let us denoted by* $\overline{P}$ *and* $\underline{P}$ *be the upper and lower conditional probabilities defined by the outer and inner Hausdorff measures as in Theorem 2. Then two events A and B of* $\mathbf{G}^0$*, s-independent and with Hausdorff dimension less then 1, are logical independent.*

Proof. Since $\dim_H$(A) and $\dim_H$(B) are both less then 1 if A and B are s-independent then the following conditions hold:

1) $\dim_H(AB)=\dim_H(B)=\dim_H(A)$

2) $\underline{P}(A|B)=\underline{P}(A|B^c)=\underline{P}(A)=\underline{h}^1(A)=0$ and $\underline{P}(B|A)=\underline{P}(A|A^c)=\underline{P}(B)=\underline{h}^1(B)=0$

3) $\overline{P}(A|B)=\overline{P}(A|B^c)=\overline{P}(A)=\overline{h}^1(A)=0$ and $\overline{P}(B|A)=\overline{P}(B|A^c)=\overline{P}(B)=\overline{h}^1(B)=0$.

From 1) we have that $A\cap B\neq \emptyset$ since the Hausdorff dimension of the empty set can not be equal to that one of any other set, from 3) we have $\overline{P}(A|B)=0$ then B is not contained in A and $\overline{P}(B|A)=0$ then A is not contained in B. Moreover since $\dim_H A$ and $\dim_H B$ are both less then 1 then $h^1(A\cup B)=0$ while $h^1(\Omega)=1$ so $\Omega \neq A\cup B$. □

We prove that logical independence is a necessary condition for the s-independence when the events have Hausdorff dimension equal to 1 and positive and finite Lebesgue outer measure.

Theorem 5 *Let Ω=[0,1], let **F** be the σ-field of all subsets of [0,1], let **G**=**F** and let us denoted by $\overline{P}$ and $\underline{P}$ be the upper and lower conditional probabilities defined by the outer and inner Hausdorff as in Theorem 2. Two events A and B of $\mathbf{G}^0$, s-independent, with Hausdorff dimension equal to 1 and such that $0<\overline{h}^1(A)<1$ and $0<\overline{h}^1(B)<1$, are logically independent.*

Proof. Since A and B are s-independent, from condition 1 we have $\dim_H A\cap B=1$, that implies $A\cap B\neq \emptyset$; from condition 3 we have $\overline{P}(A|B)=\overline{P}(A|B^c)=\overline{P}(A)=\overline{h}^1(A)\neq 1$ so B is not contained in A and B^c is not contained in A; moreover $\overline{P}(B|A)=\overline{P}(B|A^c)=\overline{P}(B)=\overline{h}^1(B)\neq 1$ so A is not contained in B and A^c is not contained in B.Then A and B are logically independent. □

We can observe that the converse of Theorems 3 and 5 is not true; in fact logical independence is not a sufficient condition for the s-independence.

Example 3 *Let Ω=[0,1], let **F** be the σ-field of all subsets of [0,1], let **G**=**F** and let us denoted by $\overline{P}$ and $\underline{P}$ be the upper and lower conditional probabilities defined by the outer and inner Hausdorff measures as in Theorem 2. Let A and B two finite subsets of [0,1] such that each of four sets $A\cap B$, $A\cap B^c$, $A^c\cap B$, $A^c\cap B^c$ is non-empty, that is A and B are logical independent. We have that A and B are not s-independent since conditions 2 and 3 of Definition 2 are never satisfied.*

If **G** is properly contained in **F** and A belong to **F**-**G**, for any H in $\mathbf{G}^0$ we cannot test the s-independence between A and H because epistemic independence is symmetric, so it requires that also A belongs to $\mathbf{G}^0$; in this case we introduce the following definition.

Definition 2. Let Ω=[0,1], let **F** be the σ-field of all subsets of [0,1] and let **G** a sub field of **F**. Denoted by $\overline{P}$ and $\underline{P}$ be the upper and lower conditional probabilities defined by the outer and inner Hausdorff measures and given A in F and B in $\mathbf{G}^0$, then B is *s-irrelevant* to A if the following conditions hold:

1) $\dim_H(AB)=\dim_H(B)=\dim_H(A)$

2) $\underline{P}(A|B)=\underline{P}(A|B^c)=\underline{P}(A)$ and $\overline{P}(A|B)=\overline{P}(A|B^c)=\overline{P}(A)$.

Proposition 1 *Let Ω=[0,1], let **F** be the σ-field of all subsets of [0,1] and **G** a sub field properly contained in **F**. Given A in **F** and B in $\mathbf{G}^0$ such that $dim_H(A)<1$, $dim_H(B)<1$ and B is s-irrelevant to A then the following conditions hold:*
1a) $A\cap B\neq\emptyset$;
2a) B is not contained in A and B^c is not contained in A;
3a) $\Omega\neq A\cup B$;

Proof. The result follows from Theorem 5. □

Definition 3. Let Ω=[0,1], let **F** be the σ-field of all subsets of [0,1] and let **G** an additive subclass contained in **F**. Given A in **F** we say that **G** is *s-irrelevant* to A if any event H of **G** such that $\dim_H$(A)=$\dim_H$(H) is irrelevant to A.

The previous results can be used to solve paradoxical situations proposed in literature that show that the interpretation of conditional probability in terms of partial knowledge breaks down in certain cases. A conditional probability can be used to represent partial information as proposed by Billingsley [1]. A probability space (Ω,**F**,P) can be use to represent a random phenomenon or an experiment whose outcome is drawn from Ω according to the probability given by P. Partial information about the experiment can be represented by a sub σ-field **G** of **F** in the following way: an observer does not know which ω has been drawn but he knows for each H in **G**, if ω belongs to H or if ω belongs to H^c.

A sub σ-field **G** of **F** can be identified as partial information about the random experiment, and, fixed A in **F**, conditional probability can be used to represent partial knowledge about A given the information on **G**. By standard definition, an event A is independent from the σ-field **G** if it is independent from each H in **G**, that is, if conditional probability is defined by the Radon-Nikodym derivative, $P[A||\mathbf{G}]_\omega$=P(A) with probability 1. Example 3 shows that the interpretation of conditional probability in terms of partial knowledge breaks down in certain cases. In fact the event A is independent from the information represented by **G** and this is a contradiction according to the fact that the information represented by **G** is complete since **G** contains all the singletons of Ω. The contradiction can be dissolved if s-irrelevance is tested with respect to conditional probabilities assigned by a class of Hausdorff dimensional measures.

Example 4 *Let (Ω,**F**,P) be a probability space where Ω=[0,1], **F** is the σ-field of Borel of Ω and P is the Lebesgue measure on **F**. Let **G** be the sub σ-field of **F** of sets that are either countable or co-countable. Let $\overline{P}$ be the finitely additive conditional probability defined on $\mathbf{C}=\mathbf{F}\times\mathbf{G}^0$ by*

$$\overline{P}(A|H)=\begin{cases}\frac{h^1(AH)}{h^1(H)} & H \quad \textit{co-countable}\\ \frac{h^0(AH)}{h^0(H)} & H \quad \textit{finite}\\ m(AH) & H \quad \textit{countable}\end{cases} \tag{3}$$

where m is a 0-1 valued finitely additive (but not countably additive) probability *measure.*

Given A=[a,b] with 0<a<b<1, we have that **G** is not s-irrelevant to A, since condition 2 of the definition of s-irrelevance is not satisfied.

In fact for every H= [0,1]-{ω} we have that $\overline{P}$(A)=$\overline{P}$(A|Ω)=h^1(A) is different from 0 and 1, while P*(A|H^c)=P*(A|{ω}) must be, for the coherence, equal to 1 or 0 according to the fact that ω belongs to A or not.

5 Conclusions and Applications

The results proposed in this paper would be an attempt to show that Hausdorff dimensional measures can be used as a tool to define coherent conditional probabilities. This approach is based on the idea that *commensurable* events [4] with respect to the given coherent conditional probability, are subsets of Ω with the same Hausdorff dimension. Given a coherent conditional probabilities P* defined on **C** =**F**×**G**^{0}, any pair of events A and B of **G**^{0} can be compare as proposed by de Finetti. In fact

$$P^*(A|A\cup B)+P^*(B|A\cup B)\geq 1$$

so the above conditional probabilities cannot be both zero and their ratio can be used to introduce an ordering between A and B. In fact this ratio is finite if either P*(A|A∪B) and P*(B|A∪B) are finite and in this case A and B are called *commensurable*. Otherwise if one of the conditional probability is zero the corresponding event has a probability infinitely less then the other and the two events A and B belong to different layers [5]. We can observe that when conditional probability P* is countably additive there can be only finitely many layers above a given layer, but not so when P is only finitely additive.

Two events A and B of **G**^{0}, commensurable with respect to the coherent conditional probability defined by (3) of Example 4, are subsets of Ω with the same Hausdorff dimension. The converse is not true, in fact if A is countable and B finite then the two events have Hausdorff dimension equal to 0, but they are not commensurable with respect to the previous conditional probability, since coherence requires that P*(B|A∪B)=0. Two events are commensurable in the sense of de Finetti if and only if they have both finite and positive Hausdorff measure and the same Hausdorff dimension.

Also from a practical point of view there are some advantages to assess coherent conditional probabilities, by a class of Hausdorff dimensional measures. In fact they can be used as a tool to assess probability to an event given a data set coming from a real problem. In different fields of science (geology, biology, architecture, economics) many data sets are fractal sets, i.e. are sets with non-integer Hausdorff dimension; for example the hypocentre distribution of earthquakes is a fractal set so if we want to assess the probability that a given place will be the hypocentre of a future earthquake knowing the set of the previous ones, we need to have a tool able to handling fractal sets. Moreover the classification of several soils can be done by their Hausdorff dimensions. A future aim of this research is to implement these results to dealing uncertainty in natural hazard and risk assessment.

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