How to Deal with Partially Analyzed Acts? A Proposal

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Abstract

In some situations, a decision is best represented by an incompletely analyzed act: conditionally to a certain event, the consequences of the decision on sub-events are perfectly known and uncertainty becomes expressable through probabilities, whereas the plausibility of this event itself remains vague and the decision outcome on the complementary event is imprecisely known. In this framework, we study an axiomatic decision model and prove a representation theorem. Decision criteria must aggregate partial evaluations consisting in: i) the conditional expected utility associated with the analyzed part of the decision and ii) the best and worst outcomes of its non-analyzed part.

Keywords

decision making under uncertainty, partially analyzed decision

1 Introduction

Consider the famous oil wildcatter problem of decision analysis textbooks. Its description only involves geophysical data and results of seismic tests, which makes it quite convincingly expressible in a Savagean setting where decisions are acts and events are endowed with subjective probabilities. However, it may well be that the relevance of that analysis is only contingent on local political stability. A complete description of the problem would require introducing this factor explicitly. The likelihood of political events being generally difficult to assess and their impact on the wildcatter profits difficult to evaluate, the standard Savagean approach reveals itself unsuitable for taking this aspect into account.

As another example, consider the question of the use of genetically modified organisms (GMO) in agriculture. Without GMO, farmers' income depend basically on climatic and market variables. Available data allow to estimate their probability distribution and their impact on income. With GMO, expected income remains assessable conditionally on the absence of cross-fertilization and contamination of other plants. However, neither the plausibility of the contamination, nor its consequences on the farmers' income, can be precisely evaluated. Here again, the standard approach appears unsatisfactory.

In these situations, and many others (introduction of new technologies, marketing of new medicines,...) decisions seem best represented by incompletely analyzed acts: conditionally to some events consequences of decisions on sub-events are perfectly known and uncertainty becomes expressable through probabilities, whereas the plausibility of these events themselves remains vague and the decision outcomes on complementary events are imprecisely known.

The axiomatic model proposed below is an attempt at formalizing such situations and at justifying adapted decision criteria.

2 The Model

2.1 Decisions

Consider: Ω, set of states of nature; $\mathcal{E}$, $\sigma-$algebra of events; $\mathcal{C}$, a set of consequences; $\mathcal{G}$, $\sigma-$algebra of subsets of $\mathcal{C}$ containing singletons. A decision problem involves a particular set of decisions, $\mathcal{D}$, which are (measurable) acts in the sense of Savage[1], i.e., mappings $(\Omega,\mathcal{E}) \longrightarrow (\mathcal{C},\mathcal{G})$. However, in the decision model below, these acts are not completely known by the decision maker. Specifically, the decisions are only partially analyzed, i.e., for any decision $a \in \mathcal{D}$ there is an event A such that the restriction of a to A - the analyzed part of a - denoted $a|_A$ is exactly known but the only information about $a|_{A^c}$ - the non-analyzed part of a - is its range $M_a = a(A^c)$. Thus, preferences will depend on pairs $(a|_A, M_a)$.

A specific feature of the model is that $\mathcal{D}$ is not assumed to contain all conceivable pairs $(a|_A, M_a)$. The reason is that decision makers cannot be expected to meaningfully evaluate unrealistic decisions. Thus the range M on an "unfavorable" event (such as a natural catastrophe) should not include any blissful consequence. Similarly, in some situations, major ignorance about the relevant event will necessarily imply much uncertainty about outcomes i. e. a wide consequence range M on this event.

Completely analyzed decisions, denoted by $(a|_\Omega, \cdot)$, can exist. In particular, for evaluation purposes, we shall assume the existence of completely analyzed $\mathcal{R}$-measurable acts, where subalgebra $\mathcal{R}$ of $\mathcal{E}$ can be interpreted as events associated with sequences of heads and tails (see Savage [6, p. 38-39] and de Finetti [2,

[1]More precisely, we use Savage's remark [6, § 3.4, p. 42] that the results in his model remain valid with events, consequences and acts defined in the present way.

p.199-202]).

A decision a analyzed on an event A is called an A - act. It generates a σ - algebra of subsets of A : $\left\{ a|_A^{-1}(G),\ G \in \mathcal{G} \right\}$, which we embed into a richer one, the σ - algebra $\mathcal{A}_a$ of subsets of A generated by $\left\{ a|_A^{-1}(G) \cap R,\ G \in \mathcal{G},\ R \in \mathcal{R} \right\}$.

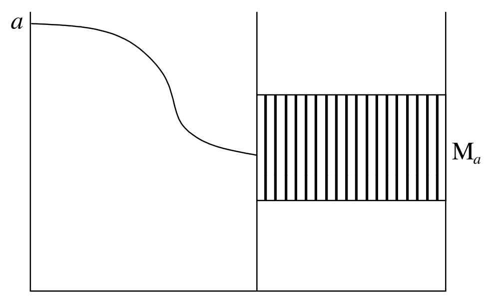

Figure 1: A partially analyzed act

We denote by $\mathcal{F}_a$ the set of all pairs $g = (g|_A, M_a)$ where $g|_A$ is any conceivable Savagean act $(A, \mathcal{A}_a) \longrightarrow (\mathcal{C}, \mathcal{G})$. Thus, $g \in \mathcal{F}_a$ implies $M_g = M_a$. There is one such set corresponding to each $a \in \mathcal{D}$ and their union is denoted by $\mathcal{F}$. We denote by $\mathcal{A}_{\mathcal{F}}$ the set of all events A such that $\mathcal{F}$ contains at least one A-act.

Note that the fact that two acts a' and a'' are both A-acts, i.e., are analyzed on the same event A, does not imply the identity of $\mathcal{A}_{a'}$ and $\mathcal{A}_{a''}$, nor that of $\mathcal{F}_{a'}$ and $\mathcal{F}_{a''}$.

Example 1 *Acts a, a', a'' characterize various oil field management strategies in the same country. Political risk (event A^c) may imply partial or complete loss of the investment. Act a' involves the same investment level I as a but concerns the exploitation of a different oil field, whereas act a'' corresponds to a more intensive exploitation of the same field as a. Thus, it is likely that $M_{a'} = M_a = [0, -I]$ but $\mathcal{A}_{a'} \neq \mathcal{A}_a$ (oil yields depend on different events), whereas $M_{a''} = [0, -I''] \neq M_a$ and $\mathcal{A}_{a''} = \mathcal{A}_a$. Hence, although the three acts are analyzed on the same event A, $\mathcal{F}_a$, $\mathcal{F}_{a'}$ and $\mathcal{F}_{a''}$ all differ from one another.*

2.2 Preferences

Preferences on $\mathcal{F}$ are expressed by a binary relation $\succsim$. We assume:

Axiom 1 *$\succsim$ is a weak order on $\mathcal{F}$.*

We want to endow $\succsim$ with standard properties and, moreover, to establish links between its restrictions $\succsim_a$ to the various $\mathcal{F}_a$. For this, we need in particular an appropriate version of Savage's Sure Thing Principle.

Due to the partial information on the decisions, the common part $Com(a,b)$ of two acts a and b analyzed on events A and B, respectively, is defined as

$$Com(a,b) = \begin{cases} \{\omega \in A \cap B : a(\omega) = b(\omega)\} \text{ if } M_a \neq M_b \\ \{\omega \in A \cap B : a(\omega) = b(\omega)\} \cup (A^c \cap B^c) \text{ if } M_a = M_b \end{cases}$$

Axiom 2 *(Sure Thing Principle for partially analyzed decisions)*

Let $a, \widehat{a}, b, \widehat{b} \in \mathcal{F}$ where $\widehat{a}$ results from a and $\widehat{b}$ from b by a common modification in the sense that $Com(a,b) = Com(\widehat{a}, \widehat{b})$.

Then $a \succsim b \Longleftrightarrow \widehat{a} \succsim \widehat{b}$.

Note that the feasible common modifications of a given pair of acts are strongly limited by the fact that the modified acts must still belong to $\mathcal{F}$.

Note also that $\mathcal{F}_a$, $\mathcal{F}_{\widehat{a}}$, $\mathcal{F}_b$, $\mathcal{F}_{\widehat{b}}$ may differ.

Example 2 *Suppose there are three countries: $\mathbb{A}, \mathbb{B}$ and $\mathbb{C}$. Country $\mathbb{A}$ (resp. $\mathbb{B}$) may possibly face an economic crisis (event A^c (resp. B^c)) which however is unlikely in country $\mathbb{C}$. A firm has to take a decision concerning a productive investment of amount I. The decision a of investing I in country $\mathbb{A}$ will generate sales shared out among countries $\mathbb{A}, \mathbb{B}$ and $\mathbb{C}$ in proportions 45% in country $\mathbb{A}$, 5% in country $\mathbb{B}$ and 50% in country $\mathbb{C}$, unless economic crisis (event A^c) happens in $\mathbb{A}$ in which case I may be partially or completely lost, independently of crisis occurring or not in country $\mathbb{B}$.*

On the other hand, consider a' with the same amount of investment in $\mathbb{A}$ as a but generating a different sales sharing, namely 70%, 30% and 0% respectively in countries $\mathbb{A}, \mathbb{B}$ and $\mathbb{C}$ if there is no economic crisis. With this investment decision, the firm may loose up to I if crisis occurs only in $\mathbb{A}$, but is sure to loose the investment completely if the crisis takes place simultaneously in $\mathbb{A}$ and $\mathbb{B}$ (event $A^c \cap B^c$).

Decisions b and b' have similar characteristics with the roles of countries $\mathbb{A}$ and $\mathbb{B}$ exchanged. We assume moreover that the countries are "similar", in the sense that the return from sales is the same in $\mathbb{A}$ as in $\mathbb{B}$, that is $a|_A = c$ and $b|_B = c$ with $c \in \mathcal{C}$.

Thus, a and b are respectively an $A-$act and a $B-$act with

$$Com(a,b) = (A \cap B) \cup (A^c \cap B^c)$$

and $M_a = M_b = [0, -I]$.

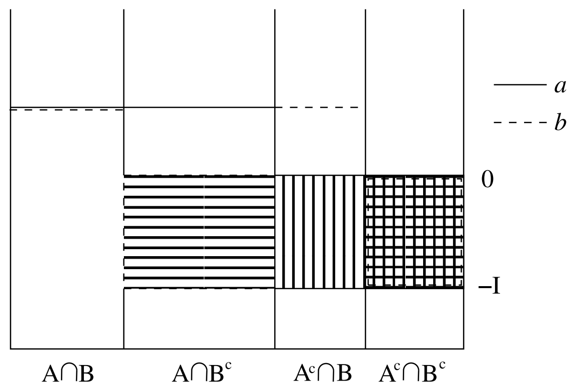

Figure 2: Original acts a and b.

a' and b' are $(A \cap B) \cup (A^c \cap B^c)$ − acts resulting from a and b by a modification of their common part. More precisely,

$$a|_{A\cap B} = b|_{A\cap B} = a'\big|_{A\cap B} = b'\big|_{A\cap B},$$
$$M_{a'} = M_{b'} = [0, -I] \text{ and } a'\big|_{A^c\cap B^c} = b'\big|_{A^c\cap B^c} = -I$$

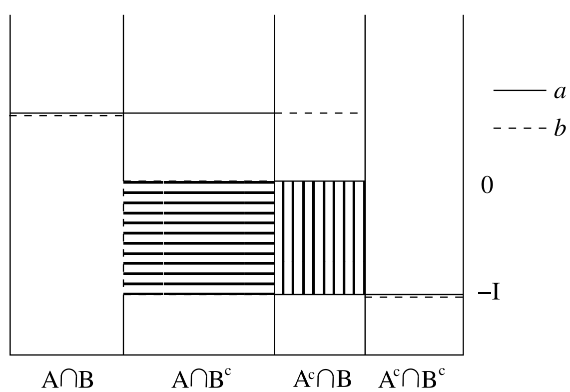

Figure 3: Modified acts a' and b'.

3 Preferences on Analyzed Events and SEU

From preferences $\succsim_a$ on $\mathcal{F}_a$, we can now derive, "à la Savage", $\succsim_a^E$, conditional preferences given event E, where $E \in \mathcal{A}_a$, by

$$g \succsim_a^E h \Leftrightarrow g' \succsim_a h' \text{ where } g'\big|_E = g|_E,\ h'\big|_E = h|_E \text{ and } g'\big|_{A\setminus E} = h'\big|_{A\setminus E}$$

Axiom 2 ensures that the ordering of g' and h' is independent from their common values on $A\backslash E$.

Note that $\succsim_a^A$ is the same as $\succsim_a$.

More generally, given an A-act, $a \in \mathcal{D}$ and a B-act, $b \in \mathcal{D}$ where $B \in \mathcal{A}_a$ (hence $B \subset A$), orderings $\succsim_a^B$ on $\mathcal{F}_a$ and $\succsim_b$ on $\mathcal{F}_b$ are related as shown by the following lemma.

Lemma 1 *Let $a' = (a'|_A\,,\, M_a)\,,\; a'' = (a''|_A\,,\, M_a)$ with $a', a'' \in \mathcal{F}_a$. Suppose that, for some $B \in \mathcal{A}_a$, b is a $B-$act and $b' = (a'|_B\,,\, M_b)\,, b'' = (a''|_B\,,\, M_b) \in \mathcal{F}_b$. Then*

$$a' \succsim_a^B a'' \Leftrightarrow b' \succsim_b b''.$$

Proof. Consider g' and g'' resulting from b' and b'' by the common modification consisting in giving them a constant common consequence $g'(\omega) = g''(\omega) = c$ for $\omega \in A\backslash B$ and the same range M_a on A^c. By Axiom 2, $g' \succsim g'' \Leftrightarrow b' \succsim_b b''$. Moreover, g' and g'' also belong to $\mathcal{F}_a$ and can be obtained by modifying a' and a'' on $A\backslash B$ and giving them the constant value c. By definition, $a' \succsim_a^B a'' \Leftrightarrow g' \succsim g''$. Hence $a' \succsim_a^B a'' \Leftrightarrow b' \succsim_b b''$. □

As a direct consequence of Lemma 1, conditional preferences given E are intrinsic in the sense that they do not depend on which $\mathcal{A}_a$ containing E (hence on which a in $\mathcal{F}$) is considered, and can be defined by $g \succsim^B h \Leftrightarrow$ there is a such that $g \succsim_a^E h$.

We also need slightly modified versions of the other Savage's definitions and axioms.

A constant A-act f_a^c in $\mathcal{F}_a$ is defined by: $f_a^c(A) = \{c\}$ with $c \in \mathcal{C}$ and $f_a^c(A^c) = M_a$.

Savage's P3 becomes:

Axiom 3 *For c', $c'' \in \mathcal{C}$, let $f_a^{c'}$, $f_a^{c''}$ be constant A-acts in $\mathcal{F}_a$ and $f_b^{c'}$, $f_b^{c''}$ be constant B-acts in $\mathcal{F}_b$.*

Then $f^{c'} \succsim f^{c''} \Longleftrightarrow f_b^{c'} \succsim f_b^{c''}$.

Preferences among consequences can now be defined by

$c' \succeq_{\mathcal{C}} c'' \Longleftrightarrow$ there exist $a \in \mathcal{D}$ and constant A-acts $f_a^{c'}$, $f_a^{c''}$ in $\mathcal{F}_a$ such that $f_a^{c'} \succsim f_a^{c''}$.

Since $\mathcal{C}$ can always be replaced by its quotient, we henceforth assume w.l.o.g. that $\succeq_{\mathcal{C}}$is an order (i.e. is antisymmetric) which justifies the use of symbol $\succeq_{\mathcal{C}}$.

We moreover assume the existence of $\underline{c}$, $\overline{c}$, respectively the worst and the best consequence in $\mathcal{C}$.

We now require Savage's P4 (irrelevance of the values of the prizes on the events) in every $\mathcal{F}_e$, where $e \in \mathcal{D}$ is an E-act.

Axiom 4 *Let $A, B \in \mathcal{A}_e$ where $e \in \mathcal{D}$ is an E-act; let $c_1, c_1', c_2, c_2' \in \mathcal{C}$ be such that $c_1 \succ_C c_1'$ and $c_2 \succ_C c_2'$. Define acts $f, f', g, g' \in \mathcal{F}_e$ by:*
i) $f(E^c) = f'(E^c) = g(E^c) = g'(E^c) = M_e$;
ii) $f(\omega) = c_1, \quad f'(\omega) = c_1'$, for $\omega \in A$;
$f(\omega) = c_2, \quad f'(\omega) = c_2'$, for $\omega \in E \backslash A$;
iii) $g(\omega) = c_1, \quad g'(\omega) = c_1'$, for $\omega \in B$;
$g(\omega) = c_2, \quad g'(\omega) = c_2'$, for $\omega \in E \backslash B$;
then $f \succsim g \Leftrightarrow f' \succsim g'$.

Whenever $f \succsim g$ holds for $f,\ g$ defined as in Axiom 4, we can write $A \succsim_e^E B$. However, if $A, B \in \mathcal{A}_{e^*}$ for some other $e^* \in \mathcal{D}$ which is also an $E-$act, it results from Axiom 2 ($f(E^c) = f'(E^c) = g(E^c) = g'(E^c) = M_e$ above can be replaced by $f(E^c) = f'(E^c) = g(E^c) = g'(E^c) = M_{e^*}$) that: $A \succsim_e^E B \Leftrightarrow A \succsim_{e^*}^E B$. We can therefore drop the subscript e and simply write $A \succsim^E B$ and read "event A is qualitatively more probable than B conditionally to event E".

The next axiom is Savage's P5.

Axiom 5 *There exists a pair $c', c'' \in \mathcal{C}$ such that $c' \succ_C c''$.*

We also introduce a version of Savage's P6. It makes it clear that one of the roles of the coin-toss related subalgebra of events $\mathcal{R}$ is to make all $(\mathcal{A}_a, \succsim_a)$ atomless.

Axiom 6 *Let $f, g \in \mathcal{F}_a$, where $a \in \mathcal{D}$ is an A-act, with $f \succ g$ and $c \in \mathcal{C}$. There exists a partition of A, consisting of events $R \cap A$, $R \in \mathcal{R}$, such that if f (resp. g) is modified on any element of the partition and given constant outcome c on this element, then the modified act f' (resp. g') also satisfies $f' \succ g$ (resp. $f \succ g'$).*

We also need Savage's P7 for each $\succsim_a$.

Axiom 7 *Let $f, g \in \mathcal{F}_a$, where $a \in \mathcal{D}$ is an A-act and let $E \in \mathcal{A}_a$. If $f \succsim_a^E$ (resp. $\precsim_a^E$) $g(\omega)$ for all $\omega \in E$, then $f \succsim_a^E$ (resp. $\precsim_a^E$) g.*

Axioms 1-7 imply the validity of Savage's P1-7 in every $\mathcal{F}_a$, where thus his main result holds: preferences in $\mathcal{F}_a$ can be represented by a subjective expected utility (SEU) criterion with respect to an atomless probability on $\mathcal{A}_a$.

Moreover, due to the explicit introduction of σ-algebra $\mathcal{R}(A) = \{A \cap R, R \in \mathcal{R}\}$ in the statement of Axiom 6, it is clear that this result still holds if $\mathcal{F}_a$ is replaced by its restriction to $\mathcal{R}(A)$ - measurable acts. We can thus state:

Proposition 1 *For every $a \in \mathcal{D}$ there exist a bounded utility u_a and an additive probability P_a such that*

$$f \succsim g \Leftrightarrow \int_A u_a \circ f \, dP_a \geq \int_A u_a \circ g \, dP_a, \quad \forall f, g \in \mathcal{F}_a$$

where

u_a is unique up to an affine transformation;

P_a is unique and for every $\rho \in [0,1]$ there exists $B \in \mathcal{A}_a$ such that $P_a(B) = \rho$.

Moreover, these existence and uniqueness statements are also valid when $\mathcal{F}_a$ is replaced by its restriction to $\mathcal{R}(A)$ - measurable acts and thus $\mathcal{A}_a$ by $\mathcal{R}(A)$.

4 Intrinsic Utility and Probability Consistency

It is well known that Savage's axioms do not imply the existence of certainty equivalents for the acts. However, this property is easily acceptable for sufficiently rich consequence sets (for instance when $\mathcal{C}$ is a real interval) and, although not necessary, will be technically helpful later in the paper. So, we assume:

Axiom 8 *For any $a \in \mathcal{F}$ there exist $c \in \mathcal{C}$ such that the constant A-act f_a^c satisfies $f_a^c \sim_a a$*

The next assumption and the lemma that follows assert that coin-toss related events are "qualitatively" independent and thus "quantitatively" independent from events in $\mathcal{E}$.

Axiom 9 *For every $A, B \in \mathcal{A}_{\mathcal{F}}$ conditional preferences on events $\succsim^A$ and $\succsim^B$ satisfy, for all $R', R'' \in \mathcal{R}$:*

$$A \cap R' \succsim^A A \cap R'' \Longleftrightarrow B \cap R' \succsim^B B \cap R''.$$

Lemma 2 *Let $a, b \in \mathcal{D}$. For every $R \in \mathcal{R}$, $P_a(A \cap R) = P_b(B \cap R)$.*

Proof. For any $R', R'' \in \mathcal{R}$, $P_a(A \cap R') \geq P_a(A \cap R'') \Leftrightarrow A \cap R' \succsim^A A \cap R'' \Longleftrightarrow B \cap R' \succsim^B B \cap R'' \Leftrightarrow P_b(B \cap R') \geq P_b(B \cap R'')$. Thus, the mapping $\mathcal{R}(A) \longmapsto [0,1]$ defined by $A \cap R \longmapsto P_b(B \cap R)$ is a probability measure representing $\succsim^A$ which however is uniquely represented by P_a. Therefore $P_a(A \cap R) = P_b(B \cap R)$ for every $R \in \mathcal{R}$. $\square$

Whenever $A \cap R' \succsim^A A \cap R''$ holds for $R', R'' \in \mathcal{R}$ and some $A \in \mathcal{A}_{\mathcal{F}}$, we shall simply write $R' \succsim^{\mathcal{R}} R''$ and read "event R' is qualitatively more probable than R'' ". Qualitative probability $\succsim^{\mathcal{R}}$ is uniquely represented by probability $P_{\mathcal{R}}$ defined by $P_{\mathcal{R}}(R) = P_a(A \cap R)$ for some A.

Thus, Axiom 8 ensures the existence of an intrinsic probability $P_{\mathcal{R}}$ on $\mathcal{R}$.

We shall use this result to derive properties of utilities. That far, all we know about the $u_a, a \in \mathcal{D}$ is that they represent the same ordering $\succeq_C$ and are therefore increasing transforms from one another. We would like functions u_a to be identical (after calibration).

According to Proposition 1 for every triple $c' \succeq_C c \succeq_C c''$, with $c' \succ_C c''$, there is an event $R \in \mathcal{R}$ such that act $g \in \mathcal{F}_a$ with $g(\omega) = c'$, for $\omega \in A \cap R$, and $g(\omega) = c''$ for $\omega \in A \cap R^c$ is indifferent to the constant $A-$act f_a^c in $\mathcal{F}_a$. In other terms, there is $R \in \mathcal{R}$ such that $P_a(A \cap R)$ satisfies:

$$u_a(c) = P_a(A \cap R)u_a(c') + (1 - P_a(A \cap R))u_a(c''), \tag{1}$$

hence, according to the definition that follows Lemma 2

$$u_a(c) = P_{\mathcal{R}}(R)u_a(c') + (1 - P_{\mathcal{R}}(R))u_a(c'').$$

Thus, all we need is an axiom ensuring that the event R in (1) only depends on c.

Axiom 10 *For every triple $c' \succeq_C c \succeq_C c''$, with $c' \succ_C c''$, there exist an event $R \in \mathcal{R}$ such that for every $a \in \mathcal{D}$, act $g \in \mathcal{F}_a$ with $g(\omega) = c'$, for $\omega \in A \cap R$, and $g(\omega) = c''$ for $\omega \in A \cap R^c$ is indifferent to the constant $A-$act f_a^c in $\mathcal{F}_a$.*

If follows immediately that:

Proposition 2 *Utilities u_a ($a \in \mathcal{D}$) are affine transforms from one another.*

Thus, after calibration u_a's are identical and we will write from now on u instead of u_a. Note that u is a utility function representing $\succeq_C$.

Next proposition guarantees the existence of intrinsic conditional probabilities in the sense that they are independent from the context in which they are evaluated.

Proposition 3 *Let $a, b \in \mathcal{D}$ be analyzed on A and B, respectively, with $B \in \mathcal{A}_a$ and let moreover $E \in \mathcal{A}_b$ (hence $E \subset B \subset A$). Then $P_a(E/B) = P_b(E)$.*

Proof. By Proposition 2, there exists $R \in \mathcal{R}$ such that $R \cap B \sim_b E$, and thus, by Lemma 1, $R \cap B \sim_a^B E$, implying

$$P_b(R \cap B) = P_b(E) \text{ and } P_b(R \cap B/B) = P_a(E/B). \tag{2}$$

Moreover, by applying Lemma 1 to acts offering prizes on events $R' \cap B$ and $R'' \cap B$, where $R', R'' \in \mathcal{R}$, we get $R' \cap B \succsim_b R'' \cap B \Leftrightarrow R' \cap B \succsim_a^B R'' \cap B$. Thus, the same ordering (say $\succsim_b$) on set of events $\{R \cap B,\ R \in \mathcal{R}\}$ is representable by (restrictions of) probabilities P_b and $P_a(./B)$; by uniqueness of such a representation (see Proposition 2), $P_b(R \cap B) = P_a(R \cap B/B)$, for all $R \in \mathcal{R}$. Then according to (2) $P_b(E) = P_a(E/B)$. □

Thus, as for conditional preferences, intrinsic conditional probabilities can be defined by $P(E/B) = P_a(E/B)$ where E, $B \in \mathcal{A}_a$ and $E \subset B$.

5 Preferences on Non-analyzed Events

We now turn to the non-analyzed part of the decisions.

Let $\mathcal{M}$ denote the set of ranges corresponding to all the decisions:

$\mathcal{M} = \{M_a,\ \exists\, a \in \mathcal{D}$ such that $a = (a|_A\,,\ M_a)\}$.

We assume that every $M \in \mathcal{M}$ has a $\succeq_C$ - greatest and a $\succeq_C$ - lowest consequence, respectively denoted $g(M)$ and $l(M)$.

We define a partial preference relation over $\mathcal{M}$. For this, two axioms are needed: Axiom 11 ensures the existence of the relation and Axiom 12 its transitivity.

Axiom 11 *Let a', a'' be $A-$acts such that $a' = (a'|_A\,,\ M_{a'})$, $a'' = (a''|_A\,,\ M_{a''})$ with $a'|_A = a''|_A$ and let b', b'' be $B-$acts such that $b' = (b'|_B\,,\ M_{b'})$, $b'' = (b''|_B\,,\ M_{b''})$ with $b'|_B = b''|_B\,,\ M_{b'} = M_{a'}$ and $M_{b''} = M_{a''}$. Then*

$$a' \succsim a'' \Leftrightarrow b' \succsim b''.$$

Preferences among ranges can now be defined by the transitive closure $\succsim_{\mathcal{M}}$ of the relation $\succsim^0_{\mathcal{M}}$ given by:

$M' \succsim^0_{\mathcal{M}} M'' \Longleftrightarrow$ there exist $A-$ acts $a', a'' \in \mathcal{D}$ such that $M_{a'} = M'$, $M_{a''} = M''$, $a'|_A = a''|_A$ and $a' \succsim a''$.

$\succsim_{\mathcal{M}}$ is automatically a partial order if:

Axiom 12 *$\succsim^0_{\mathcal{M}}$ is acyclic i.e. there is no sequence M^i, $i = 1..n$ in $\mathcal{M}$ such that $M^i \succsim^0_{\mathcal{M}} M^{i+1}$, $i = 1..n-1$ and $M^n \succ^0_{\mathcal{M}} M^1$.*

Let's now turn to the representation of the preference relation $\succsim_{\mathcal{M}}$.

The following requirement will allow us to extend a result of Barbera, Barrett and Pattanaik [1].

Axiom 13 *(1) $\forall M, c, \exists\, A$ and two $A-$acts a', a'' such that $M_{a'} = M$ and $M_{a''} = M \cup \{c\}$*

(2) Let $c_1, c_2 \in \mathcal{C}$ be such that $c_1 \succ_C c_2$. Then, for any $M_0 \in \mathcal{M}$ such that $c_1, c_2 \notin M_0$,

$$\{c_1\} \cup M_0 \succsim_{\mathcal{M}} \{c_1, c_2\} \cup M_0 \succsim_{\mathcal{M}} \{c_2\} \cup M_0.$$

Moreover, if $c \succ_C c_2$ for all $c \in M_0$, then:

$$\{c_1\} \cup M_0 \succ_{\mathcal{M}} \{c_1, c_2\} \cup M_0$$

and if $c_1 \succ_C c$ for all $c \in M_0$, then

$$\{c_1, c_2\} \cup M \succ_{\mathcal{M}} \{c_2\} \cup M_0.$$

Note that, if $M_0 = \emptyset$, we get

$$\{c_1\} \succ_{\mathcal{M}} \{c_1, c_2\} \succ_{\mathcal{M}} \{c_2\}.$$

Note that Axiom 16 makes both existence and comparability requirements.

Lemma 3 *(i) For all finite $M \in \mathcal{M}$ such that $g(M) \succ_C l(M)$, $M \sim_{\mathcal{M}} \{g(M), l(M)\}$.*
(ii) For finite $M', M'' \in M$:

$$\left.\begin{array}{r} g(M') \succeq_C g(M'') \\ l(M') \succeq_C l(M'') \end{array}\right\} \Rightarrow M' \succsim_{\mathcal{M}} M''. \tag{3}$$

Moreover,

$$\left.\begin{array}{r} g(M') \succ_C g(M'') \\ l(M') \succ_C l(M'') \end{array}\right\} \Rightarrow M' \succ_{\mathcal{M}} M''. \tag{4}$$

Proof. *(i)* For $c \in M \backslash \{g(M), l(M)\}$, by Axiom 13, $g(M) \succ_C c$ implies $M \backslash \{c\} \succsim_{\mathcal{M}} M$ (take $M_0 = M \backslash \{g(M), c\}$) and symmetrically $c \succ_C l(M)$ implies $M \succsim_{\mathcal{M}} M \backslash \{c\}$; hence $M \sim_{\mathcal{M}} M \backslash \{c\}$.

Let $M = \{g(M), c_1, c_2, ..., c_n, l(M)\}$ where $g(M) \succ_C c_1 \succ_C c_2 \succ_C ... \succ_C c_n \succ_C l(M)$. Then, by repeated application of last relation:

$M \sim_{\mathcal{M}} M \backslash \{c_1\} \sim_{\mathcal{M}} M \backslash \{c_1, c_2\} \sim_{\mathcal{M}} ...$

$\sim_{\mathcal{M}} M \backslash \{c_1, c_2, ..., c_n\} = \{g(M), l(M)\}$.

(ii) From (i) of the Lemma, we have $M' \sim_{\mathcal{M}} \{g(M'), l(M')\}$ and $M'' \sim_{\mathcal{M}} \{g(M''), l(M'')\}$. $\succsim_{\mathcal{M}}$ being transitive (Axiom 12), we just need to prove that $\{g(M'), l(M')\} \succsim_{\mathcal{M}} \{g(M''), l(M'')\}$. Assume that in the left side of (3) there is at least one strict preference, for instance $g(M') \succ_C g(M'')$ (if it is not the case, the result is straightforward). By Axiom 13 (point (2)) with $M_0 = \{l(M')\}$, we have $\{g(M'), l(M')\} \succsim_{\mathcal{M}} \{g(M''), l(M')\}$. If $l(M') \succ_C l(M'')$, by the same Axiom with $M_0 = \{g(M'')\}$, $\{g(M''), l(M')\} \succsim_{\mathcal{M}} \{g(M''), l(M'')\}$. Else $(l(M') \sim_C l(M''))$, from the proof of (i)

$$\left\{g(M''), l(M'), l(M'')\right\} \sim_{\mathcal{M}} \left\{g(M''), l(M')\right\} \sim_{\mathcal{M}} \left\{g(M''), l(M'')\right\}.$$

The proof of the second part of (ii) is similar and uses the second part of point (2) in Axiom 13 (strict inequalities). □

Lemma 3 directly implies that, for a finite sequence $(M_i)_{i=1}^n$ of finite M_i with $g(M_i)$ and $l(M_i)$ independent of i, $\cup_{j=1}^n M_j \sim_{\mathcal{M}} M_i$, $i = 1..n$. We extend this property to infinite unions in the following axiom.

Axiom 14 *For any family $(M_i)_{i \in I}$, of finite $M_i \in \mathcal{M}$ such that $g(M_i)$ and $l(M_i)$ are independent of i, $\cup_{j \in I} M_j \sim_{\mathcal{M}} M_i$, $i \in I$.*

We can then prove the following proposition:

Proposition 4 *For all $M \in \mathcal{M}$ such that $g(M) \succ_{\mathcal{C}} l(M)$, $M \sim_{\mathcal{M}} \{g(M), l(M)\}$.*

Proof. It is sufficient to note that any M in $\mathcal{M}$ is the infinite union of finite subsets of it also in $\mathcal{M}$ and with the same greatest and lowest elements. □

Proposition 5 *There exists a mapping $v : \mathcal{M} \to \mathbb{R}$ such that*

$$\begin{aligned} M' \succ_{\mathcal{M}} M'' &\Rightarrow v(M') > v(M'') \\ M' \sim_{\mathcal{M}} M'' &\Rightarrow v(M') = v(M'') \end{aligned}$$

with $M \mapsto v(M) = \varphi(g(M), l(M))$ and

$$\left.\begin{array}{c} g(M') \succ_{\mathcal{C}} g(M''), l(M') \succeq_{\mathcal{C}} l(M'') \\ or \\ g(M') \succeq_{\mathcal{C}} g(M''), l(M') \succ_{\mathcal{C}} l(M'') \end{array}\right\} \Rightarrow v(M') > v(M'').$$

Proof. Let the elements of $\mathcal{C}$ be indexed as $c_1 \succ_{\mathcal{C}} c_2 \succ_{\mathcal{C}} \ldots \succ_{\mathcal{C}} c_N$ and mapping φ defined:

$$\begin{aligned} &\text{for } i < j \text{ by } \varphi(c_i, c_j) = \sum_{(r,s)\in E_{ij}} \frac{1}{2^{r+s}}, \\ &\text{where } E_{ij} = \left\{(r,s) : r < s \text{ and } \left\{c_i, c_j\right\} \succ_{\mathcal{M}} \{c_r, c_s\}\right\} \\ &\text{for } i = j \text{ by } \varphi(c_i, c_i) = \sum_{(r,s)\in F_i} \frac{1}{2^{r+s}}, \\ &\text{where } F_i = \{(r,s) : r < s \text{ and } \{c_i\} \succ_{\mathcal{M}} \{c_r, c_s\}\} \end{aligned}$$

Then, v defined by $v(M) = \varphi(g(M), l(M))$ has the required properties since if $g(M) \succ_{\mathcal{M}} l(M)$ then $M \sim_{\mathcal{M}} \left\{c_i, c_j\right\}$ for some $c_i = g(M)$ and $c_j = l(M)$ and if $g(M) = l(M)$ $M \sim_{\mathcal{M}} \{c_i\}$ for $c_i = g(M)$. □

6 Representation Theorem

We now want to construct a utility representation of preferences $\succsim$ in $\mathcal{F}$ that incorporates the results obtained so far concerning its restrictions $\succsim_a$ to the various $\mathcal{F}_a$ as well as those concerning $\succsim_{\mathcal{M}}$.

This construction will be based on the existence of certainty equivalent for the acts which is directly required by the following axiom, where f^k denotes the constant act: $f^k(\Omega) = \{k\}$.

Axiom 15 *For any act $a \in \mathcal{F}$ there exists $k \in \mathcal{C}$ such that $f^k \sim a$.*

Proposition 6 *The weak order $\succsim$ on $\mathcal{F}$ is representable by a utility function V :*

- *For an A-act a such that $A \neq \Omega$,*

$$a = (\,a|_A\,,\, M_a) \longmapsto V(a) = \Phi\left(A,\, \int_A u \circ a\, dP_a,\, g(M_a),\, l(M_a)\right)$$

where

P_a is a subjective conditional probability on the σ-algebra $\mathcal{A}_a$;
$g(M_a)$, $l(M_a)$ are the $\succeq_{\mathcal{C}}$ - greatest and the $\succeq_{\mathcal{C}}$ - lowest consequences in M_a; and Φ is increasing in $\int_A u \circ a\, dP_a$, $g(M_a)$, $l(M_a)$.

- *Otherwise, for $A = \Omega$,*

$$a = (\,a|_\Omega\,,\,\cdot\,) \longmapsto V(a) = \Psi\left(\int_\Omega u \circ a\, dP_a\right)$$

with Ψ increasing in $\int_\Omega u \circ a\, dP_a$.

Proof. Any a in $\mathcal{F}$ has a certainty equivalent k in $\mathcal{C}$ (by Axiom 15) and $\succeq_{\mathcal{C}}$ is representable by utility function u. A priori consequence k, hence number $u(k)$, depends on all the elements characterizing a namely A, $\mathcal{A}_a$, $a|_A$ and $\mathcal{A}_a$.

Since, by Axiom 8, there exist c in $\mathcal{C}$ such that $a \sim_a f^c_a$, then

$$a \sim (A,\, \mathcal{A}_a,\, f^c_a\big|_A\,,\, M_a). \tag{5}$$

The constant A-act f^c_a being measurable with respect to any σ-algebra $\mathcal{A}_a$ of subsets of A, we have, for any A-acts a', a'' such that $M_{a'} = M_{a''}$ and $f^c_{a'} = f^c_{a''}$, $a' \sim a''$. Thus, the preference between a' and a'' does not explicitly depend on $\mathcal{A}_{a'}$ and $\mathcal{A}_{a''}$ and (5) becomes:

$$a \sim (A,\, f^c_a\big|_A\,,\, M_a). \tag{6}$$

Moreover, the certainty equivalent k depends on $a|_A$ only through $\int_A u \circ a\, dP_a$ (by Proposition 1) and on M_a only through $g(M_a)$, $l(M_a)$ (by Proposition 4). □

Example 3 *A common practice in international borrowing consists in classifying countries into various groups according to their insolvency risk. The rating is generally based on a check-list of economic indicators through a multiple criteria decision model; probability evaluations are rarely involved (Cf: Saini and Bates [5]). A given country is then allowed to borrow money at an interest rate* *equal to the LIBOR, i plus a risk spread Δi, which depends on its group. Thus, the net expected present value of a one period investment I is*

$$EV = -I + \frac{ER}{(1+i+\Delta i)} = -I + \frac{ER}{(1+i)} - \frac{\Delta i \times ER}{(1+i+\Delta i)}; \tag{7}$$

the risk premium, given by the last term, is proportional to the expected return ER. On the contrary, a particular, additive, instance of our model would evaluate the preceding investment according to formula:

$$V^* = -I + \frac{ER}{(1+i)} - k \times I$$

i. e. require the risk premium to be proportional to the maximal possible loss, here I, which seems to make more sense.

7 Discussion

The family of criteria described by the representation theorem is still rather wide and various behavioural assumptions could be added and lead to more specific criteria. On the other hand, the building blocks of the model, SEU for the analyzed part and “(max, min)” for the non-analyzed one could easily be replaced by other theories for instance the analyzed part would still be be endowed with probabilities but Quiggin’s Rank Dependent Utility [4] would replace EU or information on the non-analyzed part of the acts would not be quantified in terms of consequence sets but according to symbolic categories.

The model is consistent with various generalizations of SEU. For instance partially analyzed acts are a special case of multivalued acts; once restricted to this special class, the criteria of Ghirardato’s model [3], become a subfamily of ours. Moreover, our model allows the expression of various types of beliefs concerning the relative plausibility of the analyzed and the non analyzed events ranging from probabilities ($P(A) + P(A^c) = 1$) to complete ignorance that include capacities ($v(A) + v(A^c) \neq 1$), and in particular necessities (for instance $N(A) = \alpha, N(A^c) = 0$).

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