The Shape of Incomplete Preferences*

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Abstract

The emergence of robustness as an important consideration in Bayesian statistical models has led to a renewed interest in normative models of incomplete preferences represented by imprecise (set-valued) probabilities and utilities. This paper presents a simple axiomatization of incomplete preferences and characterizes the shape of their representing sets of probabilities and utilities. Deletion of the completeness assumption from the axiom system of Anscombe and Aumann yields preferences represented by a convex set of state-dependent expected utilities, of which at least one must be a probability/utility pair. A strengthening of the state-independence axiom is needed to obtain a representation purely in terms of a set of probability/utility pairs.

Keywords

axioms of decision theory, Bayesian robustness, state-dependent utility, coherence, partial order, imprecise probabilities and utilities

1 Introduction

In the Bayesian theory of choice under uncertainty, a decision maker holds rational preferences among acts, which are mappings from states of nature $\{s\}$ to consequences $\{c\}$. It is typically assumed that rational preferences are *complete*, meaning that for any two acts $\mathbf{X}$ and $\mathbf{Y}$, either $\mathbf{X} \succsim \mathbf{Y}$ ("$\mathbf{X}$ is weakly preferred to $\mathbf{Y}$) or else $\mathbf{Y} \succsim \mathbf{X}$, or both. This assumption, together with other rationality axioms such as transitivity and independence, leads to a representation of preferences by a unique subjective probability distribution on states $p(s)$ and a unique utility function $u(c)$ on consequences, such that $\mathbf{X} \succsim \mathbf{Y}$ if and only if the subjective expected utility of $\mathbf{X}$ is greater than or equal to that of $\mathbf{Y}$ (Savage 1954, Anscombe and Aumann 1963, Fishburn 1982). However, the completeness assumption may be inappropriate if we have only partial information about the decision maker's preferences, or if realistic limits on her powers of discrimination are assumed, or if there are actually many decision makers whose preferences may disagree.

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Incomplete preferences are generally represented by indeterminate (i.e., set-valued) probabilities and/or utilities. Varying degrees of such indeterminacy have been modeled previously in the literature of statistical decision theory and rational choice:

i. If probabilities alone are considered to be indeterminate, then preferences can be represented by a set of probability distributions $\{p(s)\}$ and a unique (perhaps linear) utility function $u(c)$. The set of probability distributions is typically convex, so the representation can be derived by separating hyperplane arguments (e.g., Smith (1961), Suppes (1974), Williams (1976), Giron and Rios (1980), Nau (1992).) Representations of this kind are are widely used in robust Bayesian statistics; an extensive treatment is given by Walley (1991).

ii. If utilities alone are considered to be indeterminate, preferences can be represented by a set of utility functions $\{u(c)\}$ and a unique (perhaps objectively specified) probability distribution $p(s)$, a representation that has been axiomatized and applied to economic models by Aumann (1962). The set of utility functions in this case is also typically convex, so that separating hyperplane arguments are again applicable.

iii. If both probabilities and utilities are allowed to be indeterminate, they can be represented by separate sets of probability distributions $\{p(s)\}$ and utility functions $\{u(c)\}$ whose elements are paired up arbitrarily. This representation of preferences preserves the traditional separation of information about *beliefs* from information about *values* when both are indeterminate (Rios Insua 1990, 1992), but lacks a natural axiomatic basis. Rather, it arises only as a special case of more general representations when probability and utility assessments are carried out independently.

iv. More generally, we can represent incomplete preferences by sets of probability distributions paired with state-independent utility functions $\{(p(s), u(c))\}$, a.k.a. "probability/utility pairs." This representation has an appealing multi-Bayesian interpretation and provides a normative basis for techniques of robust decision analysis (Moskowitz, Preckel and Yang, 1993) and asset pricing in incomplete financial markets (Staum 2002). It has been axiomatized by Seidenfeld, Schervish, and Kadane (1995, henceforth SSK), starting from the "horse lottery" formalization of decision theory introduced by Anscombe and Aumann (1963). However, as pointed out by SSK, the set of probability/utility pairs is typically nonconvex and may even be unconnected, so that separating hyperplane arguments are not directly applicable. Instead, SSK rely on methods of transfinite induction and indirect reasoning to obtain their results.

The objective of this paper is to derive a simple representation of incomplete preferences for the elementary case of finite state and reward spaces, and to characterize the shape of the resulting sets of probabilities and utilities. We begin by deleting both completeness and state-independence from the horse-lottery axiom system of Anscombe and Aumann, showing that this leads immediately to a representation of preferences by a set of probabilities paired with state-*dependent* utility functions $\{(p(s), u(s,c))\}$. Such pairs will be called *state-dependent expected utility* (s.d.e.u.) functions. State-dependent utilities have been used in economic models by Karni (1985) and Drèze (1987) and are also discussed by Schervish et al. (1990). A set of s.d.e.u. functions is typically convex—unlike a set of probability/utility pairs—so that separating-hyperplane methods are still applicable at this stage. We then re-introduce Anscombe and Aumann's state-independence assumption and show that it imposes (only) the further requirement that the representing set should contain *at least one* probability/utility pair. Finally, we consider the additional assumptions that must be imposed in order to shrink the representation to (the convex hull of) a set of probability/utility pairs, and present a constructive alternative to SSK's indirect reasoning method. We show that although the representing set of probability/utility pairs is nonconvex, it nonetheless has a simple configuration: it is merely the intersection of a convex set of s.d.e.u. functions with the nonconvex surface of state-independent utilities.

The organization of the paper is as follows. Section 2 introduces basic notation and derives a representation of preferences by convex sets of s.d.e.u. functions when neither completeness nor state-independence is assumed. Section 3 incorporates Anscombe and Aumann's state-independence assumption and shows that it requires (only) the existence of at least one agreeing state-independent utility. Section 4 discusses an example of SSK to highlight the implications of different continuity and strictness conditions. Section 5 gives the additional constructive axiom that is needed to obtain a representation purely in terms of probability/utility pairs, illustrated by another example. Section 6 briefly discusses the results.

2 Representation of incomplete preferences

Let S denote a finite set of states and let C denote a finite set of consequences. Let $\mathcal{B} = \{\mathbf{B} : S \times C \mapsto \Re\}$. An element $\mathbf{X} \in \mathcal{B}$ is a *horse lottery* if $\mathbf{X} \geq \mathbf{0}$ and $\forall s,\ \sum_c X(s,c) = 1$, with the interpretation that $X(s,c)$ is the objective probability of receiving consequence c when state s occurs. Henceforth, the symbols $\mathbf{W}$, $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$, and $\mathbf{H}$ will be used to denote horse lotteries; the symbol $\mathbf{B}$ will denote an element of $\mathcal{B}$ that is not necessarily a horse lottery (e.g., $\mathbf{B}$ may represent the difference between two horse lotteries). A horse lottery $\mathbf{X}$ is *constant* if the probabilities it assigns to consequences are constant across states—i.e., if $X(s,c) = X(s',c)$ for all s, s', c. The symbol $\succsim$ will denote non-strict preference between horse lotteries: $\mathbf{X} \succsim \mathbf{Y}$ means that $\mathbf{X}$ is preferred or indifferent to $\mathbf{Y}$, which is considered as the behavioral primitive. The domain of $\succsim$ is the set of all horse lotteries. The asymmetric part of $\succsim$ will be denoted by $\succ$.

An *event* is a subset of S. The symbol $\mathbf{E}$ will be used interchangeably as the name for an event and for its indicator function on $S \times C$. That is, $E(s,c) = 1[0]$ for all c if the event $\mathbf{E}$ includes [does not include] state s. $\mathbf{E}_s$ will denote the indicator vector for state s. That is, $\mathbf{E}_s(s',c) = 1$ for all c if $s = s'$ and zero otherwise. If α is a scalar between 0 and 1, then $\alpha\mathbf{X} + (1-\alpha)\mathbf{Y}$ is an *objective mixture* of $\mathbf{X}$ and $\mathbf{Y}$: it yields consequence c in state s with probability $\alpha X(s,c) + (1-\alpha)Y(s,c)$. If $\mathbf{E}$ is an event, then $\mathbf{EX} + (\mathbf{1}-\mathbf{E})\mathbf{Y}$ is a *subjective mixture* of $\mathbf{X}$ and $\mathbf{Y}$: it yields consequence c in state s with probability $X(s,c)$ if $E(s,c) = 1$, and with probability $Y(s,c)$ otherwise.

Assume that C contains a "worst" and a "best" consequence, labeled 0 and 1 respectively.[1] Other consequences are labeled $2,3,\ldots,K$. The symbols $\mathbf{H}_c$, for $c \in \{0,1,2,\ldots,K\}$, and $\mathbf{H}_u$, for $u \in (0,1)$, will be used to denote special "reference" horse lotteries. First, for all $c \in \{0,1,2,\ldots,K\}$, let $\mathbf{H}_c$ denote the horse lottery that yields consequence c with probability 1 in every state. That is, $\mathbf{H}_c(s,c') = 1$ if $c = c'$ and $\mathbf{H}_c(s,c') = 0$ otherwise. For example, $\mathbf{H}_2$ is the horse lottery that yields consequence 2 with probability 1 in every state. Next, for all $u \in (0,1)$, let $\mathbf{H}_u$ denote the horse lottery that yields the best and worst consequences with probabilities u and $1-u$ in every state, which is the objective mixture:

$$\mathbf{H}_u \equiv u\mathbf{H}_1 + (1-u)\mathbf{H}_0.$$

For example, $\mathbf{H}_{0.5}$ is the horse lottery that yields consequences 0 and 1 with equal probability. Later on, consequences 0 and 1 will be assigned utilities of 0 and 1, respectively, so that $\mathbf{H}_u$ will have an expected utility of u by definition.

The reference-lottery notation can be stretched further to define $\mathbf{H}_\mathbf{E}$ as the horse lottery that yields the best consequence if event $\mathbf{E}$ occurs and the worst consequence otherwise, i.e., the subjective mixture:

$$\mathbf{H}_\mathbf{E} \equiv \mathbf{E}\mathbf{H}_1 + (\mathbf{1}-\mathbf{E})\mathbf{H}_0.$$

Bounds on subjective probabilities are expressible as preferences between subjective and objective mixtures of $\mathbf{H}_0$ and $\mathbf{H}_1$. For example, a preference of the form $\mathbf{H}_\mathbf{E} \succsim \mathbf{H}_p$ for some event $\mathbf{E}$ and $p \in (0,1)$ means that "the probability of $\mathbf{E}$ is at least p," i.e., that p is a *lower probability* for $\mathbf{E}$. Upper probabilities are defined analogously. If $\mathbf{X}$ is a horse lottery and u is a scalar between 0 and 1, a preference

[1] Our assumption of *a priori* best and worst consequences follows Luce and Raiffa (1957) and Anscombe and Aumman (1963), and it is technically without loss of generality in the sense that the preference order can always be extended to a larger domain that includes two additional consequences which by construction are better and worse, respectively, than all the original consequences. (Such an extension is demonstrated by SSK, Theorem 2.) The best and worst consequences ultimately serve to calibrate the definition and measurement of subjective probabilities, but even so the probabilities remain somewhat arbitrary, as will be shown.

of the form $\mathbf{X} \succsim \mathbf{H}_u$ means that "the expected utility of $\mathbf{X}$ is at least u." Equivalently, we will say that u is a *lower expected utility* for $\mathbf{X}$. Upper expected utilities are defined analogously. Using the terms defined above, we now introduce the first group of axioms that are assumed to govern rational preference:

A1 (Quasi order): $\succsim$ is transitive and reflexive.

A2 (Mixture-independence): $\mathbf{X} \succsim \mathbf{Y} \Leftrightarrow \alpha\mathbf{X} + (1-\alpha)\mathbf{Z} \succsim \alpha\mathbf{Y} + (1-\alpha)\mathbf{Z} \quad \forall\alpha \in (0,1)$.

A3 (Continuity in probability): If $\{\mathbf{X}_n\}$ and $\{\mathbf{Y}_n\}$ are convergent sequences such that $\mathbf{X}_n \succsim \mathbf{Y}_n$, then $\lim \mathbf{X}_n \succsim \lim \mathbf{Y}_n$.

A4 (Existence of best and worst): For all $c > 1$, $\mathbf{H}_1 \succsim \mathbf{H}_c \succsim \mathbf{H}_0$.

A5 (Coherence, or non-triviality): $\mathbf{H}_1 \succ \mathbf{H}_0$ (i.e., *not* $\mathbf{H}_0 \succsim \mathbf{H}_1$).

A1 and A2 are von Neumann and Morgenstern's first two axioms of expected utility, minus completeness[2], as applied to horse lotteries by Anscombe and Aumann (1963); see also Fishburn (1982). A3 is a strong continuity condition used by Garcia del Amo and Rios Insua (2002) that also works in infinite-dimensional spaces. A4 and A5 ensure non-triviality and provide reference points for probability measurement, as noted earlier.

DEFINITION: A collection of preferences $\{\mathbf{X}_n \succsim \mathbf{Y}_n\}$ is a *basis*[3] for $\succsim$ under an axiom system if every preference $\mathbf{X} \succsim \mathbf{Y}$ can be deduced from $\{\mathbf{X}_n \succsim \mathbf{Y}_n\}$ by direct application of those axioms.

The primal geometric representation of $\succsim$ is now given by:

Theorem 1 *$\succsim$ satisfies A1–A5 if and only if there exists a closed convex cone $\mathcal{B}^* \subset \mathcal{B}$, receding from the origin, such that for any horse lotteries $\mathbf{X}$ and $\mathbf{Y}$:*

$$\mathbf{X} \succsim \mathbf{Y} \Leftrightarrow \mathbf{X} - \mathbf{Y} \in \mathcal{B}^*.$$

In particular, if $\{\mathbf{X}_n \succsim \mathbf{Y}_n\}$ is a basis for $\succsim$ under A1–A5, then the cone $\mathcal{B}^*$ is the closed convex hull of the rays whose directions are $\{\mathbf{X}_n - \mathbf{Y}_n\}$ for all n together with $\{\mathbf{H}_1 - \mathbf{H}_c\}$ and $\{\mathbf{H}_c - \mathbf{H}_0\}$ for all c.[4]

Because the direction of preference between two horse lotteries $\mathbf{X}$ and $\mathbf{Y}$ depends only on the direction of the vector $\mathbf{X} - \mathbf{Y}$, it follows that if $\mathbf{EX} + (\mathbf{1} - \mathbf{E})\mathbf{Z} \succsim \mathbf{EY} + (\mathbf{1} - \mathbf{E})\mathbf{Z}$ where $\mathbf{E}$ is an event, then $\mathbf{EX} + (\mathbf{1} - \mathbf{E})\mathbf{Z}' \succsim \mathbf{EY} + (\mathbf{1} - \mathbf{E})\mathbf{Z}'$ for any $\mathbf{Z}'$. Consequently, we will simply write $\mathbf{EX} \succsim \mathbf{EY}$ to indicate that $\mathbf{EX} + (\mathbf{1} - \mathbf{E})\mathbf{Z} \succsim \mathbf{EY} + (\mathbf{1} - \mathbf{E})\mathbf{Z}$ for all $\mathbf{Z}$, or in other words, "$\mathbf{X}$ is preferred to $\mathbf{Y}$ conditional on the event $\mathbf{E}$." This result enables us to give a simple definition of conditional probability or expected utility: if $\mathbf{E}$ is an event and $\mathbf{X}$ is a horse lottery, then the preference $\mathbf{EX} \succsim \mathbf{EH}_u$ means that "the conditional expected utility of $\mathbf{X}$ given $\mathbf{E}$ is at least u."

[2] The completeness assumption asserts that for any $\mathbf{X}$ and $\mathbf{Y}$, either $\mathbf{X} \succsim \mathbf{Y}$ or $\mathbf{Y} \succsim \mathbf{X}$, or both. Here, it is permitted that neither of these conditions holds—i.e., $\mathbf{X}$ and $\mathbf{Y}$ may be incomparable.

[3] Use of the term "basis" in this context is due to SSK.

[4] Proofs have been suppressed in the conference version of the paper but are available in the complete version on the author's web site at http://www.duke.edu/∼rnau.

Now let a state dependent expected utility (s.d.e.u.) function be defined as a function $v : S \times C \mapsto \Re$, with the interpretation that $v(s,c)$ is the expected utility of receiving consequence c with probability 1 if state s obtains and receiving consequence 0 with probability 1 otherwise. Let $U_v(\mathbf{X})$ denote the expected utility assigned to a horse lottery $\mathbf{X}$ by the s.d.e.u. function v:

$$U_v(\mathbf{X}) \equiv \sum_{s \in S, c \in C} X(s,c)v(s,c).$$

DEFINITIONS: A s.d.e.u. function v is a *probability/utility pair* if it can be expressed as the product of a probability distribution on S and a state-independent utility function on C—i.e., if $v(s,c) = p(s)u(c)$ for some functions p and u. A s.d.e.u. function v *agrees* (one way) with $\succsim$ if $\mathbf{X} \succsim \mathbf{Y} \Rightarrow U_v(\mathbf{X}) \geq U_v(\mathbf{Y})$. A set $\mathcal{V}$ of s.d.e.u. functions *represents* $\succsim$ if $\mathbf{X} \succsim \mathbf{Y} \Leftrightarrow U_v(\mathbf{X}) - U_v(\mathbf{Y}) \geq 0 \; \forall \, v \in \mathcal{V}$.

We now have, as the dual to Theorem 1:

Theorem 2 *$\succsim$ satisfies A1–A5 if and only if it is represented by a non-empty closed convex set $\mathcal{V}^*$ of s.d.e.u. functions satisfying (w.l.o.g.) $U_v(\mathbf{H}_0) = 0$ and $U_v(\mathbf{H}_1) = 1$.*

(The proof relies on a separating hyperplane argument. For a similar result on a more general space, see Rios 1992.) If $\{\mathbf{X}_n \succsim \mathbf{Y}_n\}$ is a basis for $\succsim$, then $\mathcal{V}^*$ is merely the intersection of the linear constraints $\{U_v(\mathbf{X}_n) \geq U_v(\mathbf{Y}_n)\}$, $U_v(\mathbf{H}_0) = 0$, $U_v(\mathbf{H}_1) = 1$, and $0 \leq U_v(\mathbf{H}_c) \leq 1$ for all $c \geq 2$. If the basis is finite, then $\mathcal{V}^*$ is a convex polytope, whose elements need not be probability/utility pairs. Subsequent sections of the paper will discuss the additional assumptions needed to ensure that some points of $\mathcal{V}^*$—especially its extreme points—are probability/utility pairs.

3 The state-independence axiom

We now explore the implications, in the context of incompleteness, of the additional axiom introduced by Anscombe and Aumann[5] to provide the usual separation of subjective probability from utility. First, define the concept of a not-potentially-null event:

DEFINITION: An event $\mathbf{E}$ is *not potentially null* if $\mathbf{H_E} \succsim \mathbf{H}_p$ for some $p > 0$.

Thus, an event that is not potentially null is precluded from having zero as an upper probability in any extension of $\succsim$ satisfying A1–A5. The final axiom is then:

A6 (State-independence): If $\mathbf{X}$ and $\mathbf{Y}$ are constant and $\mathbf{E}$ is not potentially null, then $\mathbf{EX} \succsim \mathbf{EY} \;\Rightarrow\; \mathbf{E'X} \succsim \mathbf{E'Y}$ for every other event $\mathbf{E}'$.

An immediate contribution of A6, in light of A4, is to guarantee that consequences 0 and 1 are best and worst *in every state*. Thus, if A6 holds, any s.d.e.u.

[5] Anscombe-Aumann refer to this assumption as "monotonicity in the prizes" or "substitutability."

function agreeing with $\succsim$ may be considered to belong to the set $\mathcal{V}^+ \subset \mathcal{V}$ defined by:

$$\mathcal{V}^+ \equiv \{v : 0 = v(s,0) \leq v(s,c) \leq v(s,1) \leq 1 \; \forall\, s \in S,\; c \geq 2;\; \sum_{s \in S} v(s,1) = 1\}.$$

Henceforth it will be assumed (arbitrarily but w.l.o.g.) that consequences 0 and 1 have the same numerical utilities, namely 0 and 1, in every state as well as unconditionally. Then, regardless of whether v is a probability/utility pair, define

$$p_v(s) \equiv v(s,1)$$

as "the" probability assigned to state s by v, since it is the expected utility of a horse lottery that yields a utility of 1 if state s obtains and 0 otherwise.[6] Correspondingly, if $\mathbf{E}$ is an event,

$$p_v(\mathbf{E}) \equiv U_v(\mathbf{H_E}) = \sum_{s \in \mathbf{E}} p_v(s)$$

is the probability assigned to $\mathbf{E}$ by v. Next define:

$$u_v(s,c) \equiv v(s,c)/v(s,1) \;\text{ if }\; v(s,1) > 0,$$

as the utility assigned to consequence c in state s by v. This utility is state-independent if v is a probability/utility pair, otherwise it is state-dependent. In these terms, the expected utility assigned to $\mathbf{X}$ by v can be rewritten as:

$$U_v(\mathbf{X}) = \sum_s p_v(s) \sum_c u_v(s,c) X(s,c).$$

We can now give a dual definition of conditional expected utility in terms of v in the obvious way:

$$U_v(\mathbf{X}|\mathbf{E}) = U_v(\mathbf{XE})/p_v(\mathbf{E}).$$

If the conditional expected utility of $\mathbf{X}$ given $\mathbf{E}$ is at least u by our primal definition—i.e., if $\mathbf{EX} \succsim \mathbf{EH}_u$—then dually we have $U_v(\mathbf{X}|\mathbf{E}) \geq u$ for any v agreeing with $\succsim$ and satisfying $p_v(\mathbf{E}) > 0$, because for any agreeing v:

$$\mathbf{EX} \succsim \mathbf{EH}_u \Rightarrow U_v(\mathbf{EX}) \geq U_v(\mathbf{EH}_u) = u p_v(\mathbf{E}) \Leftrightarrow U_v(\mathbf{X}|\mathbf{E}) \geq u \;\text{ or else }\; p_v(\mathbf{E}) = 0.$$

Another consequence of A6, in light of Theorem 1, is the property of stochastic dominance. In particular, if $\mathbf{X}$ is obtained from $\mathbf{Y}$ by shifting probability mass

[6] The same method of defining probabilities is used by Karni (1993). Since this definition is based on the arbitrary assignment of equal utilities to the best and worst outcomes in all states, it should not be interpreted as the "true" probability of a hypothetical decision maker whose preferences are represented by v. The classic definitions of subjective probability given by Savage, Anscombe-Aumann, and others, are all afflicted with the same arbitrariness. The intrinsic impossibility of inferring "true" probabilities from material preferences is discussed by Kadane and Winkler (1988), Schervish et al. (1990), Karni and Mongin (2000) and Nau (1995, 2002).

to consequence 1 from any other consequence, and/or from consequence 0 to any other consequence, in any state, then $\mathbf{X} \succsim \mathbf{Y}$. To see this, note that A6 together with A4 implies that $\mathbf{E}_s\mathbf{H}_1 \succsim \mathbf{E}_s\mathbf{H}_c$ and $\mathbf{E}_s\mathbf{H}_c \succsim \mathbf{E}_s\mathbf{H}_0$ for state s and any $c > 1$. Hence $\mathcal{B}^*$ contains all vectors of the form $\mathbf{E}_s(\mathbf{H}_1 - \mathbf{H}_c)$ and $\mathbf{E}_s(\mathbf{H}_c - \mathbf{H}_0)$. If $\mathbf{X} - \mathbf{Y}$ can be expressed as a non-negative linear combination of these vectors, then $\mathbf{X} - \mathbf{Y} \in \mathcal{B}^*$ and hence $\mathbf{X} \succsim \mathbf{Y}$. To make this result more precise, let the $[\,.\,]_{\min}$ ("minimum s.d.e.u.") operation be defined on $\mathcal{B}$ as follows:

$$[\mathbf{B}]_{\min} \equiv \min_{v \in \mathcal{V}^+} U_v(\mathbf{B}) = \min_{s \in S} [B(s,1) + \sum_{c \geq 2} \min\{0, B(s,c)\}].$$

This quantity is the minimum possible state-dependent expected utility that could be assigned to $\mathbf{B}$: it is achieved by assigning, within each state, a utility of 0 to those consequences $c \geq 2$ for which $\mathbf{B}$ is positive and a utility of 1 to those consequences $c \geq 2$ for which $\mathbf{B}$ is negative, then assigning a subjective probability of 1 to the state in which the conditional expected utility of $\mathbf{B}$ is minimized. Stochastic dominance and the negative orthant in $\mathcal{B}$ can now be defined in a natural way:

DEFINITIONS: $\mathbf{X} \geq^* [>^*]\ \mathbf{Y}$ ("$\mathbf{X}$ [strictly] dominates $\mathbf{Y}$") if $[\mathbf{X} - \mathbf{Y}]_{\min} \geq [>]\ 0$. The open negative orthant $\mathcal{B}^-$ consists of those $\mathbf{B}$ that are strictly dominated by the zero vector, i.e., $\mathcal{B}^- = \{\mathbf{B} \in \mathcal{B} : \mathbf{0} >^* \mathbf{B}\}$.

A6 in conjunction with A1–A5 then implies that $\mathbf{X} \geq^* [>^*]\ \mathbf{Y} \Rightarrow \mathbf{X} \succsim [\succ]\ \mathbf{Y}$. If preferences are complete (i.e., if for any horse lotteries $\mathbf{X}$ and $\mathbf{Y}$, either $\mathbf{X} \succsim \mathbf{Y}$ or $\mathbf{Y} \succsim \mathbf{X}$ or both), then the primal representation $\mathcal{B}^*$ is a half-space, the dual representation $\mathcal{V}^*$ consists of a unique s.d.e.u. function v^*, and axiom A6 requires the latter to be a probability/utility pair, which is the same result obtained by Anscombe and Aumann (1963). (A6 implies that $U_{v^*}(\mathbf{H}_c|\mathbf{E}_s) = U_{v^*}(\mathbf{H}_c)$ independent of the state s.) In the absence of completeness, the contribution of A6 to the separation of probability and utility is weaker, as summarized by:

Theorem 3 . $\succsim$ *satisfies A1–A6 if and only if it is represented by a nonempty convex set* $\mathcal{V}^{**} \subseteq \mathcal{V}^+$ *of s.d.e.u. functions of which at least one element is a probability/utility pair.*

If $\{\mathbf{X}_n \succsim \mathbf{Y}_n\}$ is a basis for $\succsim$ under axioms A1–A6, then any probability/utility pair v that satisfies $U_v(\mathbf{X}_n) \geq U_v(\mathbf{Y}_n)$ for all n, $v \in \mathcal{V}^+$, belongs to the set $\mathcal{V}^{**}$. Apart from this fact, it is not easy to characterize the set $\mathcal{V}^{**}$ in terms of probability/utility pairs, as will be illustrated in the sequel.

4 Strict vs. non-strict preference: an example

The results of the preceding section establish that a preference relation satisfying A1–A6 is represented by a closed set of s.d.e.u. functions of which at least one is a probability/utility pair. The closedness of the representing set is attributable to the use of non-strict preference as the behavioral primitive, together with a strong continuity assumption. In contrast, SSK use strict preference as the behavioral primitive, together with a weaker continuity assumption, to explicitly allow for the representation of incomplete preferences by open sets that may fail to contain probability/utility pairs.

The differences in these approaches are illustrated by an example of SSK (Example 4.1) comprising two states and three consequences, i.e., $S = \{1,2\}$ and $C = \{0,1,2\}$. Consequences 0 and 1 have state-independent utilities of 0 and 1 by assumption, so that a probability/utility pair is completely parameterized by the probability assigned to state 1 and the utility assigned to consequence 2. Consider, then, the two probability/utility pairs (p_i, u_i) in which $p_0(1) = 0.1$ and $p_1(1) = 0.3$, and $u_0(2) = 0.1$ and $u_1(2) = 0.4$. Let v_0 and v_1 denote the corresponding s.d.e.u. functions—i.e., $v_i(s,c) = p_i(s)u_i(c)$ for $i = 0,1$. Then $U_{v_i}(\mathbf{X})$ denotes the expected utility assigned to horse lottery $\mathbf{X}$ by (p_i, u_i). In particular, $U_{v_0}(\mathbf{H}_2) = 0.1$ and $U_{v_1}(\mathbf{H}_2) = 0.4$. Now let $\succ$ be defined as the preference relation that satisfies a weak Pareto condition with respect to these two probability/utility pairs—i.e., $\mathbf{X} \succ \mathbf{Y} \Leftrightarrow \{U_{v_0}(\mathbf{X}) > U_{v_0}(\mathbf{Y})$ and $U_{v_1}(\mathbf{X}) > U_{v_1}(\mathbf{Y})\}$. Any s.d.e.u. function that is a convex combination of v_0 and v_1 also agrees with $\succ$, so the representing set $\mathcal{V}^{**}$ is the closed line segment whose endpoints are v_0 and v_1, but none of its interior points are probability/utility pairs.

Next SSK extend $\succ$ to obtain a new preference relation $\succ''$ by imposing the additional strict preferences $\mathbf{H}_{0.4} \succ'' \mathbf{H}_2 \succ'' \mathbf{H}_{0.1}$. The effect of this extension is to chop off the two endpoints of the representing set of s.d.e.u. functions, so that $\succ''$ is represented by the *open* line segment connecting v_0 with v_1. SSK point out that, although $\succ''$ satisfies all their axioms, there is no agreeing probability/utility pair for it, since the only two candidates have been deliberately excluded. They proceed to axiomatize the concept of "almost state- independent" utilities, which agree with a strict preference relation and are "within ε" of being state- independent. Clearly, $\succ''$ has an almost-state-independent representation, containing points arbitrarily close to v_0 and v_1.

In our framework, where the language of preference is non-strict, there is no way to implement a constraint such as $\mathbf{H}_2 \succ \mathbf{H}_{0.1}$ (i.e., to chop off v_0) except by asserting that $\mathbf{H}_2 \succsim \mathbf{H}_{0.1+\varepsilon}$ for a specific positive ε. And if this assertion is made, an interesting thing happens: axiom A6 begins to nibble on the v_0 end of the line segment and continues nibbling until the representation collapses to the v_1 end. To illustrate this process, let the non-zero elements of each v be written out as $v = (\{v(s,c)\}) = (v(1,1), v(2,1); v(1,2), v(2,2))$. Thus, $v_0 = (0.1, 0.9; 0.01, 0.09)$ and $v_1 = (0.3, 0.7; 0.12, 0.28)$. (Note that because these are probability/utility pairs, the first two numbers in parentheses are the probabilities of states 1 and 2, and the last two numbers are the same probabilities multiplied by the utility of consequence 2.) Next, let the line segment from v_0 to v_1 be parameterized by $v_\alpha \equiv (1-\alpha)v_0 + \alpha v_1$ for $\alpha \in (0,1)$. In these terms we obtain:

$$v_\alpha = (0.1 + 0.2\alpha, 0.9 - 0.2\alpha; 0.01 + 0.11\alpha, 0.09 + 0.19\alpha),$$

whence:

$$U_{v_\alpha}(\mathbf{H}_2) = v_\alpha(1,2) + v_\alpha(2,2) = 0.1 + 0.3\alpha \tag{4.1}$$

$$U_{v_\alpha}(\mathbf{H}_2|\mathbf{E}_1) = \frac{v_\alpha(1,2)}{v_\alpha(1,1)} = \frac{0.01 + 0.11\alpha}{0.1 + 0.2\alpha} \tag{4.2}$$

$$U_{v_\alpha}(\mathbf{H}_2|\mathbf{E}_2) = \frac{v_\alpha(2,2)}{v_\alpha(2,1)} = \frac{0.09 + 0.19\alpha}{0.9 - 0.2\alpha} \tag{4.3}$$

These are all monotone functions of α for α between 0 and 1, and they all are equal to 0.1 at $\alpha = 0$ and 0.4 at $\alpha = 1$. However, for intermediate values of α, (4.1) is greater than (4.3) and less than (4.2), and by invoking axiom A6, we can play the last two off against each other. In particular, it follows from monotonicity of (4.2) that

$$\alpha \geq \alpha^* \Rightarrow U_{v_\alpha}(\mathbf{H}_2|\mathbf{E}_1) \geq \frac{0.01 + 0.11\alpha^*}{0.1 + 0.2\alpha^*}, \tag{4.4}$$

whereas it follows from monotonicity of (4.3) that

$$U_{v_\alpha}(\mathbf{H}_2|\mathbf{E}_2) \geq u^* \Rightarrow \alpha \geq \frac{0.9u^* - 0.09}{0.2u^* + 0.19} \tag{4.5}$$

Let the set $\mathcal{V}^{**}$ representing the original relation $\succ$ henceforth be parameterized as $\mathcal{V}^{**} = \{v_\alpha | \alpha \in [0,1]\}$. Suppose that we now increase the lower utility of $\mathbf{H}_2$ by $\varepsilon = 0.01$ by adding the preference assertion $\mathbf{H}_2 \succsim \mathbf{H}_{0.11}$ to the basis for $\succ$. This additional assertion imposes the constraint $U_{v_\alpha}(\mathbf{H}_2) \geq 0.11$ for all v_α agreeing with the extended relation, thus excluding v_0 as an agreeing s.d.e.u. function. By application of A6, we may conclude that $U_{v_\alpha}(\mathbf{H}_2|\mathbf{E}_2) \geq 0.11$ as well. Substituting $u^* = 0.11$ in (4.5), it follows that the representing set must consist only of those v_α satisfying $\alpha \geq 0.042453$. But now, substituting $\alpha^* = 0.042453$ back into (4.4), we find that it must also satisfy $U_{v_\alpha}(\mathbf{H}_2|\mathbf{E}_1) \geq 0.135217$. Since $\mathbf{E}_1$ is not potentially null, A6 may be applied again to obtain $U_{v_\alpha}(\mathbf{H}_2) \geq 0.135217$. Thus, if we take one bite out of the line segment by imposing the constraint $U_{v_\alpha}(\mathbf{H}_2) \geq 0.11$, we end up concluding that a larger bite $U_{v_\alpha}(\mathbf{H}_2) \geq 0.135217$ may be taken! If we now repeat the process by substituting $u^* = 0.135217$ in (4.5), we obtain $\alpha^* = 0.146034$, which yields $U_{v_\alpha}(\mathbf{H}_2|\mathbf{E}_1) \geq 0.201721$ when substituted in (4.4). Successive iterations yield u^* values of 0.299288, 0.365247, 0.390144, 0.397381, 0.399317, and so on with rapid convergence to 0.4, which is realized (only) at v_1. The continuity axiom then allows us to assert that $\mathbf{H}_2 \succsim \mathbf{H}_{0.4}$, which together with the original constraint $\mathbf{H}_{0.4} \succsim \mathbf{H}_2$, establishes that the utility of consequence 2 is precisely 0.4.

If instead we start at the other endpoint, adding the constraint $\mathbf{H}_{0.4-\varepsilon} \succsim \mathbf{H}_2$ for $\varepsilon > 0$, the collapse occurs to the 0.1 value. If both constraints are added—i.e., if both endpoints are chopped off by finite margins, the entire interval is annihilated, yielding incoherence (a violation of A5). Hence, this example is unstable in the sense that any *finite* extension of the original preference relation leads to a collapse to one or the other of the original probability/utility pairs, or else to incoherence.

5 The need for stronger state-independence

The original preference relation in SSK’s example is represented by a set of s.d.e.u. functions whose extreme points are both probability/utility pairs. In our framework, if either of these points is excluded, then the intervening points must be excluded as well. Thus, in extending that relation, it is impossible to retain any agreeing state-dependent utilities that are not convex combinations of agreeing state-independent utilities. A second example shows that this is not always the case under axioms A1–A6. In other words, a preference relation can satisfy these axioms and yet not be represented by utilities that are state-independent or even “almost” state-independent.

Let there be three states and three consequences, and let **X** denote the horse lottery that satisfies $X(1,0) = X(2,2) = X(3,1) = 1$. That is, **X** yields consequences 0, 2, and 1 with certainty in states 1, 2, and 3 respectively. Suppose that all states are judged to have probability at least 0.1, and **X** is judged to have an unconditional expected utility of at least 0.5. Furthermore, a coin flip between **X** and {consequence 2 if state 1, otherwise **Z**} is preferred to a coin flip between utility 0.5 and {utility 0.9 if state 1, otherwise **Z**}, but also a coin flip between **X** and {utility 0.1 if state 2, otherwise **Z**} is preferred to a coin flip between utility 0.5 and {consequence 2 given state 2, otherwise **Z**}. (The common alternative **Z** is arbitrary by Theorem 1.) Thus, the basis for $\succsim$ is as follows:

$$\mathbf{H}_{\mathbf{E}} \succsim \mathbf{H}_{0.1} \ \text{ for } \ \mathbf{E} = \mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3, \tag{5.1}$$

$$\frac{1}{2}\mathbf{X} + \frac{1}{2}\mathbf{Z} \succsim \frac{1}{2}\mathbf{H}_{0.5} + \frac{1}{2}\mathbf{Z}, \tag{5.2}$$

$$\frac{1}{2}\mathbf{X} + \frac{1}{2}(\mathbf{E}_1\mathbf{H}_2 + (1-\mathbf{E}_1)\mathbf{Z}) \succsim \frac{1}{2}\mathbf{H}_{0.5} + \frac{1}{2}(\mathbf{E}_1\mathbf{H}_{0.9} + (1-\mathbf{E}_1)\mathbf{Z}), \tag{5.3}$$

$$\frac{1}{2}\mathbf{X} + \frac{1}{2}(\mathbf{E}_2\mathbf{H}_{0.1} + (1-\mathbf{E}_2)\mathbf{Z}) \succsim \frac{1}{2}\mathbf{H}_{0.5} + \frac{1}{2}(\mathbf{E}_2\mathbf{H}_2 + (1-\mathbf{E}_2)\mathbf{Z}), \tag{5.4}$$

Notice that (5.3) and (5.4) are obtained from (5.2) by replacing **Z** by subjective mixtures of **Z** with different constant lotteries on the LHS and RHS. These last two preferences imply that the lower bound on the expected utility of **X** among all *probability/utility pairs* agreeing with $\succsim$ must be strictly greater than 0.5. To understand this implication, note that under any s.d.e.u. function that agrees with $\succsim$, the differences in expected utility between the LHS’s and RHS’s of (5.2), (5.3), and (5.4), must all be non-negative. Moreover, if the s.d.e.u. function is a probability/utility pair, then in at least one of the two comparisons (5.3) and (5.4), the difference in expected utility between LHS and RHS must be strictly less than it is in (5.2), a situation that occurs when consequence 2 has a utility strictly greater than 0.1 and/or strictly less than 0.9. If the difference in expected utility between LHS and RHS is non-negative in all cases, then the difference can never be zero in (5.2)—i.e., **X** cannot have a lower expected utility as small as

0.5. In fact, the minimum expected utility of **X** among all probability/utility pairs agreeing with (5.1–5.4) is 0.564314.

The question is whether, by direct application of axioms A1–A6, we can infer that the expected utility of **X** is strictly greater than 0.5. The answer is: we cannot. The problem is that axiom A6 is useless here because of the common nonconstant term **X** in (5.2)–(5.4). In order to apply A6, we must first find non- negative linear combinations of the differences between the LHS's and RHS's of (5.1)–(5.4) that are conditionally constant—i.e., of the form **EB**, where **E** is an event and **B** is constant across states. But the search for such conditionally constant terms is constrained here by the presence of a common nonconstant term $\mathbf{X}-\mathbf{H}_{0.5}$ in the differences between LHS's and RHS's of (5.2)–(5.4). Furthermore, in order for A6 to "bite," **B** needs to have a negative lower expected utility when conditioned on some other event $\mathbf{E}'$. The effect of applying A6 will then be to raise this lower expected utility to zero, which shrinks the set of s.d.e.u. functions representing $\succsim$. In the example, the few conditionally-constant lottery differences **EB** that can be constructed from (5.1)–(5.4) all turn out to satisfy $\mathbf{B} \geq^* 0$, which is completely uninformative. The lower expected utility of **X** therefore remains at 0.5 despite the fact that this value is not realized, *or even closely approached*, by any probability/utility pair agreeing with $\succsim$.

This example shows that when preferences are incomplete, axiom A6 is insufficient to guarantee that they are represented by a set of probability/utility pairs (or their convex hull). Evidently, an additional state-independence condition is needed, such as:

A7 (Stochastic substitution): If

$$\alpha\mathbf{X}+(1-\alpha)(\mathbf{E}\mathbf{X}'+(1-\mathbf{E})\mathbf{Z}) \succsim \alpha\mathbf{Y}+(1-\alpha)(\mathbf{E}\mathbf{Y}'+(1-\mathbf{E})\mathbf{Z})$$

for some $\alpha \in (0,1)$ where $\mathbf{X}'$ and $\mathbf{Y}'$ and **Z** are constant lotteries and **E** is not potentially null, then

$$\alpha\mathbf{X}+(1-\alpha)(p\mathbf{X}'+(1-p)\mathbf{Z}) \succsim \alpha\mathbf{Y}+(1-\alpha)(p\mathbf{Y}'+(1-p)\mathbf{Z})$$

for some $p \in (0,1]$.

In other words, the subjective mixtures of the constant lotteries $\mathbf{X}'$ and $\mathbf{Y}'$ with **Z** can be replaced with objective mixtures *against the background* of a comparison between the nonconstant lotteries **X** and **Y**. In terms of the primal representation $\mathcal{B}^*$, this assumption means that if $\mathbf{B}+\mathbf{E}\mathbf{B}' \in \mathcal{B}^*$, where $\mathbf{B}'$ is constant across states and **E** is not potentially null, then $\mathbf{B}+p\mathbf{B}' \in \mathcal{B}^*$ for some $p > 0$.[7] Note that if a collection of preferences $\{\mathbf{X}_n \succsim \mathbf{Y}_n\}$ satisfies A1–A6, then the imposition of A7 cannot produce a contradiction. A1–A6 require the existence of at least one probability/utility pair agreeing with $\{\mathbf{X}_n \succsim \mathbf{Y}_n\}$, and any probability/utility pair that agrees with the original preferences will also agree with any new preferences generated from them by A7.

[7] **A2** and **A6** imply only that this substitution may be performed in the nonstochastic case $\mathbf{B}=\mathbf{0}$.

The new axiom *does* affect the counterexample discussed above. (5.3) and (5.4) can now be replaced by

$$\frac{1}{2}\mathbf{X}+\frac{1}{2}(p\mathbf{H}_2+(1-p)\mathbf{Z}) \succsim \frac{1}{2}\mathbf{H}_{0.5}+\frac{1}{2}(p\mathbf{H}_{0.9}+(1-p)\mathbf{Z}),$$

$$\frac{1}{2}\mathbf{X}+\frac{1}{2}(p'\mathbf{H}_{0.1}+(1-p')\mathbf{Z}) \succsim \frac{1}{2}\mathbf{H}_{0.5}+\frac{1}{2}(p'\mathbf{H}_2+(1-p')\mathbf{Z}),$$

for some $p, p' > 0$. A mixture of these two comparisons in a ratio of p' to p yields:

$$\frac{1}{2}\mathbf{X}+\frac{1}{2}(\alpha\mathbf{H}_{0.1}+\alpha\mathbf{H}_2+(1-2\alpha)\mathbf{Z}) \succsim \frac{1}{2}\mathbf{H}_{0.5}+\frac{1}{2}(\alpha\mathbf{H}_{0.9}+\alpha\mathbf{H}_2+(1-2\alpha)\mathbf{Z}),$$

where $\alpha = pp'/(p+p')$. The LHS must have greater-or-equal expected utility than the RHS, which (because of the $\mathbf{H}_{0.1}$ term on the left and the $\mathbf{H}_{0.9}$ term on the right, and cancellation of the common terms $\mathbf{H}_2$ and $\mathbf{Z}$) means that $\mathbf{X}$ must have strictly greater expected utility than 0.5.

The main result, which generalizes this example, can now be stated as:

Theorem 4 $\succsim$ *satisfies A1–A7 if and only if it is represented by a nonempty set* $\mathcal{V}^{***}$ *of s.d.e.u. functions that is the convex hull of a set of probability/utility pairs.*

If $\{\mathbf{X}_n \succsim \mathbf{Y}_n\}$ is a basis for $\succsim$ under A1–A7, then $\mathcal{V}^{***}$ is merely the convex hull of the set of probability/utility pairs that satisfy $\{U_v(\mathbf{X}_n) \geq U_v(\mathbf{Y}_n)\}$. If the basis is finite, the construction of $\mathcal{V}^{***}$ can be carried out as follows. First, form the convex polyhedron consisting of the intersection of the constraints $\{U_v(\mathbf{X}_n) \geq U_v(\mathbf{Y}_n)\}$, $v \in \mathcal{V}^+$. Now take the intersection of this polyhedron with the nonconvex surface consisting of all probability/utility pairs. (If the latter intersection is empty, the preferences do not satisfy A1–A7: they are incoherent.) Finally, take the convex hull of what remains: this is the set $\mathcal{V}^{***}$.

6 Discussion

It has been shown that, in order to obtain a convenient representation of incomplete preferences by sets of probability/utility pairs, it does not suffice merely to delete the completeness axiom from the standard axiomatic framework of Anscombe and Aumann. This finding is not due to technical problems with limits or null events, but rather to a fundamental weakness of the traditional state-independence axiom in the absence of completeness. Our approach is to introduce an additional state-independence postulate (A7) that has "bite" in the absence of completeness. SSK follow a different approach in their axiomatization of incomplete strict preferences. Instead of directly strengthening the state-independence property, they "fill in the missing preferences" by indirect reasoning, namely, they assume the preference relation has the property that $\neg(\mathbf{X} \succsim \mathbf{Y}) \Rightarrow \mathbf{Y} \succ \mathbf{X}$, where "$\neg$" stands for "it is precluded that," meaning that there is no extension of $\succsim$ satisfying the other axioms in which $\mathbf{X} \succsim \mathbf{Y}$ (p. 2204 ff.). SSK's assumption requires that wherever a weak preference is precluded, the opposite strict preference must be affirmed. In our framework, this property of $\succsim$ is not implied by axioms A1–A6, hence it amounts to an additional axiom of rationality. The lack of this property is illustrated by the example of the preceding section, in which it is precluded that $\mathbf{H}_u \succsim \mathbf{X}$ for any $u < 0.5643....$, yet it is not implied by A1–A6 that $\mathbf{X} \succ \mathbf{H}_u$ for any $u > 0.5$. If the "axiom" of indirect reasoning is added to A1–A6, in lieu of A7, the representation of Theorem 4 follows immediately from Theorem 3.

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