

A Sensitivity Analysis for the Pricing of European Call Options in a Binary Tree Model*

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Abstract

The European call option prices have well-known formulae in the Cox-Ross-Rubinstein model [2], depending on the volatility of the underlying asset. Nevertheless it is hard to give a precise estimate of this volatility. S. Muzzioli and C. Toricelli [6] handle this problem by using possibility distributions. In the first part of our paper we make some critical comments on their work. In the second part we present an alternative solution to the problem by performing a sensitivity analysis for the pricing of the option. This method is very general in the sense that it can be applied if one describes the uncertainty in the volatility by confidence intervals as well as if one describes it by fuzzy numbers. The conclusion is that the price of the option is not necessarily a strictly increasing function of the volatility.

Keywords

fuzzy sets, option pricing, sensitivity analysis

1 Introduction

In the first section of this paper we introduce the Cox-Ross-Rubinstein model [2] for the pricing of a European call option and the assumptions which are made. The well-known formula for the option price depends on the volatility of the underlying asset. However in practice it is hard to give a precise estimate of this volatility. S. Muzzioli and C. Toricelli [6] handle this problem by using possibility distributions. In the first part of our paper we make some critical comments on their work. In the second part we present an alternative solution to the problem

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by performing a sensitivity analysis for the pricing of the option. This method is very general in the sense that it can be applied if one describes the uncertainty in the volatility by confidence intervals as well as if one describes it by fuzzy numbers. Indeed, in both approaches the imprecise volatility results in imprecise up and down factors. Those factors are modelled by a fuzzy quantity or are said to belong to a confidence interval.

We consider the case where the down factor is the inverse of the up factor. The lifetime of the option is divided into N steps of length T/N . Then we need to study the behaviour of the option price as a function of the up factor in an interval, which is a subset of $](1+r)^{T/N}, +\infty[$, where r stands for the risk-free interest rate. Therefore we study the functional behaviour of the option price for all possible values of the up factor.

Finally, we illustrate the method by an example with a fuzzy up factor.

2 The binary tree model

The binary tree model of Cox-Ross-Rubinstein [2] can be considered as a discrete-time version of the Black & Scholes model [1]. The following assumptions are made:

- The markets have no transaction costs, no taxes, no restrictions on short sales, and assets are infinitely divisible.
- The lifetime T of the option is divided into N steps of length T/N .
- The market is complete.
- No arbitrage opportunities are allowed which implies for the risk-free rate of interest r , that $d < (1+r)^{T/N} < u$, where u is the up factor and d the down factor.

The European call option price at time zero, has a well-known formula in this model:

$$EC(K, T) = \frac{1}{(1+r)^T} \sum_{j=0}^N \binom{N}{j} p_u^j (1-p_u)^{N-j} (S_0 u^j d^{N-j} - K)_+ \quad (1)$$

where K is the exercise (or strike) price, S_0 is the price of the underlying asset at time the contract begins, p_u the risk-neutral probability that the price goes up with the factor $u = \exp(\sigma\sqrt{T/N})$, with σ the volatility of the underlying asset. Let p_d be the risk-neutral probability that the price goes down with the factor d . We assume that $d = 1/u$. It is known that p_u and p_d are solutions to the system:

$$\begin{cases} p_u + p_d = 1 \\ dp_d + up_u = (1+r)^{T/N}. \end{cases} \quad (2)$$

The solutions are:

$$p_u = \frac{(1+r)^{T/N} - d}{u - d} = \frac{(1+r)^{T/N}u - 1}{u^2 - 1} \tag{3}$$

$$p_d = \frac{u - (1+r)^{T/N}}{u - d} = \frac{u^2 - (1+r)^{T/N}u}{u^2 - 1}. \tag{4}$$

3 Critical Analysis of the paper ‘A Multiperiod Binomial Model for Pricing Options in an Uncertain World’ by S. Muzzioli and C. Torricelli

S. Muzzioli and C. Torricelli [6] state: ‘There are different methods for estimating volatility either from historical data, or from option prices. Sometimes it is hard to give a precise estimate of the volatility of the underlying asset and it may be convenient to let it take interval values. Moreover, it may be the case that not all members of the interval have the same reliability, as central members are more possible than the ones near the borders. This is exactly the idea behind our model, but instead of modelling volatility as a fuzzy quantity, we directly model the up and down jumps of the stock price.’

Instead of modelling the volatility as a fuzzy quantity, S. Muzzioli and C. Torricelli model directly the up and down factors u and d as the fuzzy numbers (u_1, u_2, u_3) and (d_1, d_2, d_3) , where u_1 (resp d_1) is the minimum possible value, u_3 (resp d_3) is the maximum possible value and u_2 (resp d_2) is the most possible value. A triangular fuzzy number (a_1, a_2, a_3) can alternatively be defined by its α -cuts $[a_1(\alpha), a_3(\alpha)]$, $\alpha \in [0, 1]$:

$$[a_1(\alpha), a_3(\alpha)] = [a_1 + \alpha(a_2 - a_1), a_3 - \alpha(a_3 - a_2)].$$

In fact a fuzzy quantity is completely defined by its α -cuts. Consider intervals $[a_1(\alpha), a_3(\alpha)]$, $\alpha \in [0, 1]$, where

$$\alpha_1 \leq \alpha_2 : [a_1(\alpha_1), a_3(\alpha_1)] \subseteq [a_1(\alpha_2), a_3(\alpha_2)],$$

then the intervals $[a_1(\alpha), a_3(\alpha)]$ are the α -cuts of the fuzzy quantity a ,

$$a(x) = \sup_{\alpha \in [0,1]} \min\{\alpha, 1_{[a_1(\alpha), a_3(\alpha)]}(x)\}, \quad x \in \mathbb{R}.$$

Since the α -cuts of a triangular fuzzy number are compact intervals of the set of real numbers, the interval calculus of Moore [5] can be applied to them. Thus every binary operation in \mathbb{R} can be extended to a binary operation on the set of fuzzy numbers.

S. Muzzioli and C. Torricelli consider a binary tree with 1 period, i.e. $T = N = 1$.

A fuzzy version of the two equations of the system (2) should be introduced. This can be done (for each equation) in two different ways:

$$\begin{aligned} p_u + p_d &= (1, 1, 1) \\ p_u &= (1, 1, 1) - p_d \end{aligned}$$

respectively

$$\begin{aligned} dp_d + up_u &= (1 + r, 1 + r, 1 + r) \\ up_u &= (1 + r, 1 + r, 1 + r) - dp_d \end{aligned}$$

where p_u and p_d are the fuzzy up and down probabilities $((p_u)_1, ((p_u)_2, ((p_u)_3)$ and $((p_d)_1, ((p_d)_2, ((p_d)_3)$.

S. Muzzioli and C. Torricelli choose for both equations the first form. However, one has to take into account that p_u and p_d are fuzzy probabilities and therefore one should use the second form for the first equation. For the second equation in (2) the first form should be taken since the left-hand side is an expectation.

Thus the correct solution is obtained by extending the system (2) to

$$\begin{cases} p_u = (1, 1, 1) - p_d \\ dp_d + up_u = (1 + r, 1 + r, 1 + r). \end{cases}$$

Expressed in α -cuts and keeping in mind that the operations are binary operations on fuzzy numbers, see e.g. E. Kerre [3], this leads to the system:

$$\begin{cases} [(p_u)_1(\alpha), (p_u)_3(\alpha)] = [1, 1] - [(p_d)_1(\alpha), (p_d)_3(\alpha)] \\ [d_1(\alpha), d_3(\alpha)][(p_d)_1(\alpha), (p_d)_3(\alpha)] + [u_1(\alpha), u_3(\alpha)][(p_u)_1(\alpha), (p_u)_3(\alpha)] \\ \quad = [1 + r, 1 + r] \end{cases}$$

or

$$\begin{cases} (p_u)_1(\alpha) = 1 - (p_d)_3(\alpha) \\ (p_u)_3(\alpha) = 1 - (p_d)_1(\alpha) \\ d_1(\alpha)(p_d)_1(\alpha) + u_1(\alpha)(p_u)_1(\alpha) = 1 + r \\ d_3(\alpha)(p_d)_3(\alpha) + u_3(\alpha)(p_u)_3(\alpha) = 1 + r. \end{cases}$$

The correct solution to this system is:

$$\begin{cases} (p_u)_1(\alpha) = \frac{d_1(\alpha)(d_3(\alpha) + u_3(\alpha)) - (1+r)(d_1(\alpha) + u_3(\alpha))}{d_1(\alpha)d_3(\alpha) - u_1(\alpha)u_3(\alpha)} \\ (p_u)_3(\alpha) = \frac{d_3(\alpha)(d_1(\alpha) + u_1(\alpha)) - (1+r)(d_3(\alpha) + u_1(\alpha))}{d_1(\alpha)d_3(\alpha) - u_1(\alpha)u_3(\alpha)} \\ (p_d)_1(\alpha) = \frac{(1+r)(d_3(\alpha) + u_1(\alpha)) - u_1(\alpha)(d_3(\alpha) + u_3(\alpha))}{d_1(\alpha)d_3(\alpha) - u_1(\alpha)u_3(\alpha)} \\ (p_d)_3(\alpha) = \frac{(1+r)(d_1(\alpha) + u_3(\alpha)) - u_3(\alpha)(d_1(\alpha) + u_1(\alpha))}{d_1(\alpha)d_3(\alpha) - u_1(\alpha)u_3(\alpha)}. \end{cases}$$

One can easily prove that for $\alpha = 1$:

$$\begin{cases} (p_u)_2 = \frac{(1+r)-d_2}{u_2-d_2} \\ (p_d)_2 = \frac{u_2-(1+r)}{u_2-d_2} \end{cases}$$

and for $\alpha = 0$:

$$\begin{cases} (p_u)_1 = \frac{d_1(d_3+u_3)-(1+r)(d_1+u_3)}{d_1d_3-u_1u_3} \\ (p_u)_3 = \frac{d_3(d_1+u_1)-(1+r)(d_3+u_1)}{d_1d_3-u_1u_3} \\ (p_d)_1 = \frac{(1+r)(d_3+u_1)-u_1(d_3+u_3)}{d_1d_3-u_1u_3} \\ (p_d)_3 = \frac{(1+r)(d_1+u_3)-u_3(d_1+u_1)}{d_1d_3-u_1u_3} \end{cases}$$

Next S. Muzzioli and C. Torricelli calculate the price of the option in the one period model. They assume that the exercise price is between the highest value of the underlying asset in state down and the lowest value of the underlying asset in state up,

$$S_0d_3 \leq K \leq S_0u_1 \tag{5}$$

in which case the calculations are very simple. The aim of their next section is to extend the pricing methodology to a two period and then to a multi period binary model. The condition (5) is extended as follows:

$$S_0d_3^{j+1}u_3^{N-j-1} \leq K \leq S_0d_1^ju_1^{N-j} \quad j = 0, \dots, N-1$$

which is impossible since K can not be an element of those N intervals. Even if one changes the condition to

$$\exists j \in \{0, \dots, N-1\} : S_0d_3^{j+1}u_3^{N-j-1} \leq K \leq S_0d_1^ju_1^{N-j}$$

the condition is not always fulfilled since one can easily prove (for example in the crisp case with $d = 1/u$) that $S_0d_3^{j+1}u_3^{N-j-1}$ is not always less than $S_0d_1^ju_1^{N-j}$. Even if this is the case, there are no economic reasons why the exercise price would not be out of the mentioned intervals.

Finally, they calculate the price of the option in one special situation of the two period model and remark that the extension to N periods is straightforward, which is not the case as we will see in what follows.

A last remark concerns the number of periods. S. Muzzioli and C. Torricelli extend the number of periods without explicitly mentioning that at the same time one should fix the lifetime of the option. Otherwise when the lifetime equals the number N of periods and N is increased, another option is considered at each step. Hence, if one models one and the same option, one has to fix the lifetime T and divide T in N subperiods of length T/N . Then increasing the number N of steps implies at the same time a decrease of the steplength.

This is also the way to proceed in order to be able to consider the important limit problem.

4 Imprecise volatility and the pricing of a European Call Option

The change of the price S_t of the underlying asset at time t can be modelled as in [4] by

$$S_{t+1} = \xi_{t+1} S_t$$

where ξ_{t+1} is a sequence taking values in a compact set M . We are interested in the special case where M consists only of two elements, its upper and lower bounds u and d . Those up and down factors depend on the volatility σ . As we already mentioned, it is often hard to give a precise estimate of the volatility. This problem can be avoided either by giving a confidence interval of the volatility or by modelling the volatility by a fuzzy quantity.

Imprecise volatility implies imprecision in the up (and down) factors. Under the assumptions of section 2 the (confidence or α -cut) intervals, to which the up factor, belongs, are subsets of $](1+r)^{T/N}, +\infty[$. We study the behaviour of the price of a European call option for all possible values of the up factor. In sections 5, 6 and 7 we also need to include the border case where the up factor equals $(1+r)^{T/N}$. Therefore we define the up factor as u_λ :

$$u_\lambda = (1+r)^{T/N} + \lambda, \quad \lambda \in \mathbb{R}^+.$$

If we invoke (3), the risk-neutral probability, p_λ , that the price goes up, is

$$p_\lambda = \frac{(1+r)^{T/N} u_\lambda - 1}{u_\lambda^2 - 1}.$$

The price $C_\lambda(K)$ of the option is:

$$\begin{aligned} C_\lambda(K) &= \frac{1}{(1+r)^T} E[(S_\lambda^T - K)_+] \\ &= \frac{1}{(1+r)^T} \sum_{j=0}^N (S_0 u_\lambda^{2j-N} - K)_+ \cdot P[X_\lambda^N = j] \\ &= \frac{1}{(1+r)^T} \sum_{j=j_\lambda^*}^N (S_0 u_\lambda^{2j-N} - K) \binom{N}{j} p_\lambda^j (1-p_\lambda)^{N-j} \end{aligned} \quad (6)$$

where X_λ^N is the number of ups in the lifetime T and $S_0 u_\lambda^{2j-N} - K$ is positive for $j \geq j_\lambda^*$.

Consider a confidence interval, $[u_0, u_1] \subset](1+r)^{T/N}, +\infty[$, of the up factor with

$$\begin{aligned} u_0 &= (1+r)^{T/N} + \lambda_0 \\ u_1 &= (1+r)^{T/N} + \lambda_1. \end{aligned}$$

If $u_\mu \in [u_0, u_1], \mu \in [0, 1]$ then

$$\begin{aligned} u_\mu &= \mu u_0 + (1 - \mu)u_1 \\ &= \mu((1+r)^{T/N} + \lambda_0) + (1 - \mu)((1+r)^{T/N} + \lambda_1) \\ &= (1+r)^{T/N} + [\mu\lambda_0 + (1 - \mu)\lambda_1] \\ &= (1+r)^{T/N} + \lambda^*(\mu). \end{aligned}$$

The price of the option belongs to the interval

$$[\min_{\mu \in [0,1]} C_{\lambda^*(\mu)}(K), \max_{\mu \in [0,1]} C_{\lambda^*(\mu)}(K)].$$

Suppose the imprecise volatility is described by using a fuzzy quantity, (u_1, u_2, u_3) , $u_1, u_2, u_3 \in (1+r)^{T/N}, +\infty[$, with

$$\begin{aligned} u_1 &= (1+r)^{T/N} + \lambda_1 \\ u_2 &= (1+r)^{T/N} + \lambda_2 \\ u_3 &= (1+r)^{T/N} + \lambda_3, \end{aligned}$$

for the up factor. An α -cut, $\alpha \in [0, 1]$, is the interval:

$$\begin{aligned} [u_1 + (u_2 - u_1)\alpha, u_3 + (u_2 - u_3)\alpha] = \\ [(1+r)^{T/N} + \lambda_1 + \alpha(\lambda_2 - \lambda_1), (1+r)^{T/N} + \lambda_3 + \alpha(\lambda_2 - \lambda_3)]. \end{aligned}$$

An element of this interval can be described by

$$\begin{aligned} \mu[(1+r)^{T/N} + (\lambda_1 + \alpha(\lambda_2 - \lambda_1))] + (1 - \mu)[(1+r)^{T/N} + \lambda_3 + \alpha(\lambda_2 - \lambda_3)] \\ = (1+r)^{T/N} + \mu(\lambda_1 + \alpha(\lambda_2 - \lambda_1)) + (1 - \mu)(\lambda_3 + \alpha(\lambda_2 - \lambda_3)) \\ = (1+r)^{T/N} + \lambda_\alpha^*(\mu), \quad \mu \in [0, 1]. \end{aligned}$$

The α -cut, $\alpha \in [0, 1]$, of the option price is:

$$[\min_{\mu \in [0,1]} C_{\lambda_\alpha^*(\mu)}(K), \max_{\mu \in [0,1]} C_{\lambda_\alpha^*(\mu)}(K)]. \quad (7)$$

It is clear that, for the method with confidence intervals as well as for the method using fuzzy quantities, the behaviour of $C_\lambda(K)$ as function of u_λ should be studied. This is the subject of the following sections.

5 Definitions, notations and lemmas

The function is broken up in its basic elements: first the (up and down) probabilities are considered, then their products and finally their products with the up and

down factors. The risk-neutral probability, p_λ , is a decreasing function of u_λ . For $u_\lambda = (1+r)^{T/N}$ this probability is one and

$$\lim_{u_\lambda \rightarrow +\infty} p_\lambda = 0.$$

And one obtains $p^* = 0.5$ for

$$u_\lambda = u^* = (1+r)^{T/N} + \sqrt{(1+r)^{2T/N} - 1}.$$

The probability $1 - p_\lambda$ is an increasing function of u_λ .

The function $p_\lambda(1 - p_\lambda)$ has a maximum for $u_\lambda = u^*$. It is zero for $u_\lambda = (1+r)^{T/N}$ and in the limit for $u_\lambda \rightarrow +\infty$.

The function $u_\lambda p_\lambda$ attains a minimum for $u_\lambda = u^*$. It is equal to $(1+r)^{T/N}$ for $u_\lambda = (1+r)^{T/N}$ and in the limit for $u_\lambda \rightarrow +\infty$.

The function $u_\lambda^{-1}(1 - p_\lambda)$ attains a maximum for $u_\lambda = u^*$. It is zero for $u_\lambda = (1+r)^{T/N}$ and in the limit for $u_\lambda \rightarrow +\infty$.

One can prove that

$$(u_\lambda p_\lambda)' = \frac{1 - 2p_\lambda}{u_\lambda^2 - 1} = -(u_\lambda^{-1}(1 - p_\lambda))'.$$

6 Functional behaviour of the functions $C_1(\lambda, j)$ and $C_2(\lambda, j, K)$

In the next section we will examine the functional behaviour of each term in the sum (6). Those terms consist of two parts, namely $C_1(\lambda, j) = S_0 u_\lambda^{2j-N} \binom{N}{j} p_\lambda^j (1 - p_\lambda)^{N-j}$ and $C_2(\lambda, j, K) = -K \binom{N}{j} p_\lambda^j (1 - p_\lambda)^{N-j}$. Those functions are first examined separately, regardless the sign of their sum.

The derivative of the function $C_1(\lambda, j)$ with respect to u_λ , $u_\lambda \in [(1+r)^{T/N}, +\infty]$, is:

$$\begin{aligned} & (C_1(\lambda, j))' & (8) \\ &= \frac{S_0 \binom{N}{j}}{u_\lambda} (u_\lambda p_\lambda)^{j-1} (u_\lambda^{-1}(1 - p_\lambda))^{N-j-1} (j(1 - p_\lambda + u_\lambda^2 p_\lambda) - N u_\lambda^2 p_\lambda) \frac{1 - 2p_\lambda}{u_\lambda^2 - 1} \end{aligned}$$

which implies that:

- If $j \leq N/2$ then $j(1 - p_\lambda + u_\lambda^2 p_\lambda) - N u_\lambda^2 p_\lambda < 0$ and the function $C_1(\lambda, j)$ attains a maximum for $u_\lambda = u^*$. It is zero for $u_\lambda = (1+r)^{T/N}$ and in the limit for $u_\lambda \rightarrow +\infty$.
- If $N/2 < j < N$ then the expression $j(1 - p_\lambda + u_\lambda^2 p_\lambda) - N u_\lambda^2 p_\lambda$

- is negative for all u_λ if moreover $(1+r)^{T/N} \leq \frac{N}{2\sqrt{(N-j)j}}$ and the function $C_1(\lambda, j)$ attains a maximum for $u_\lambda = u^*$.
 - if $(1+r)^{T/N} > \frac{N}{2\sqrt{(N-j)j}}$, $j = \text{floor}(N/2 + 1)$
 - (a) the expression is negative for $j > \frac{Nu^*}{2(1+r)^{T/N}}$ and the function $C_1(\lambda, j)$ attains a maximum for $u_\lambda = u^*$
 - (b) the expression has two roots for $j \leq \frac{Nu^*}{2(1+r)^{T/N}}$
- Those roots are:

$$u_1(j) = \frac{N + \sqrt{N^2 - 4(N-j)j(1+r)^{2T/N}}}{2(N-j)(1+r)^{T/N}}$$

$$u_2(j) = \frac{N - \sqrt{N^2 - 4(N-j)j(1+r)^{2T/N}}}{2(N-j)(1+r)^{T/N}}$$

- with $u_1(j) \geq u^* \geq u_2(j) \geq (1+r)^{T/N}$
 and if $u_1(j) = u_2(j)$ then $u_1(j) = u_2(j) = u^*$.
 The function $C_1(\lambda, j)$ attains a maximum for $u_\lambda = u_2(j)$ and for $u_\lambda = u_1(j)$. It attains a minimum for $u_\lambda = u^*$.
- The function $C_1(\lambda, j)$ is zero for $u_\lambda = (1+r)^{T/N}$ and in the limit for $u_\lambda \rightarrow +\infty$.
 - if $j = N$ then the function equals $S_0(u_\lambda p_\lambda)^N$ and it attains a minimum for $u_\lambda = u^*$. The function $C_1(\lambda, j)$ is equal to $S_0(1+r)^T$ for $u_\lambda = (1+r)^{T/N}$ and in the limit for $u_\lambda \rightarrow +\infty$.

The derivative of the function $C_2(\lambda, j, K), 0 < j < N$, with respect to u_λ is:

$$(C_2(\lambda, j, K))' = -K \binom{N}{j} (j - Np_\lambda) p_\lambda^{j-1} (1 - p_\lambda)^{N-j-1} (p_\lambda)' \tag{9}$$

The factor $(j - Np_\lambda)$ has two roots for all j :

$$u_1^*(j) = \frac{N(1+r)^{T/N} + \sqrt{N^2(1+r)^{2T/N} - 4j(N-j)}}{2j}$$

$$u_2^*(j) = \frac{N(1+r)^{T/N} - \sqrt{N^2(1+r)^{2T/N} - 4j(N-j)}}{2j}$$

but $u_2^*(j) < (1+r)^{T/N}$.
 The function attains a minimum for $u_\lambda = u_1^*(j)$. If $j \leq N/2$ then $u_1^*(j) > u^*$ and if $j \geq N/2$ then $u_1^*(j) < u^*$. The function is zero for $u_\lambda = (1+r)^{T/N}$ and in the limit for $u_\lambda \rightarrow +\infty$.
 The function is decreasing for $j = 0$. It is zero for $u_\lambda = (1+r)^{T/N}$ and equal to

$-K$ in the limit for $u_\lambda \rightarrow +\infty$.

The function $C_2(\lambda, j, K)$ increases for $j = N$. It is equal to $-K$ for $u_\lambda = (1+r)^{T/N}$ and zero in the limit for $u_\lambda \rightarrow +\infty$.

7 The branches of the binary tree considered separately

Each term in the sum corresponds to a branch in the binary tree. Such a term depends on $j, j = 0, \dots, N$, and K , and is function of λ :

$$C_\lambda(j, K) = (S_0 u_\lambda^{2j-N} - K) \binom{N}{j} p_\lambda^j (1 - p_\lambda)^{N-j}.$$

The functional behaviour of $C_\lambda(j, K)$ is examined regardless of its sign.

Noting that

$$C_\lambda(j, K) = C_1(\lambda, j) + C_2(\lambda, j, K)$$

the derivative of $C_\lambda(j, K)$ with respect to u_λ can be calculated by invoking (8) and (9):

$$\begin{aligned} & S_0 \binom{N}{j} (u_\lambda p_\lambda)^{j-1} (u_\lambda^{-1} (1 - p_\lambda))^{N-j-1} u_\lambda^{-1} (j(1 - p_\lambda + u_\lambda^2 p_\lambda) - N u_\lambda^2 p_\lambda) \frac{1 - 2p_\lambda}{u_\lambda^2 - 1} \\ & - K \binom{N}{j} (j - N p_\lambda) p_\lambda^{j-1} (1 - p_\lambda)^{N-j-1} (p_\lambda)'. \end{aligned}$$

In those intervals where both derivatives (8) and (9) have the same sign or for those values of u_λ where one of the derivatives is zero, one can immediately conclude from section 6 if the term is decreasing or increasing.

On the other hand we can draw conclusions about the functional behaviour of the term by remarking that we studied the functional behaviour of $S_0 u_\lambda^{2j-N}$ multiplied by $\binom{N}{j} p_\lambda^j (1 - p_\lambda)^{N-j}$ and that the term can be calculated in two steps: first subtract K from $S_0 u_\lambda^{2j-N}$ and then multiply the result by $\binom{N}{j} p_\lambda^j (1 - p_\lambda)^{N-j}$.

This leads to the following conclusions:

$$\boxed{j = 0}$$

- $C_0(0, K) = 0$ and $C_\lambda(0, 0) > 0$,
 $\lim_{u_\lambda \rightarrow +\infty} C_\lambda(0, K) = -K$.
- If $K \geq S_0(1+r)^{-T}$ then $C_\lambda(0, K)$ is negative for all u_λ .

- If $0 < K < S_0(1+r)^{-T}$ then $C_\lambda(0, K)$ has a root, $u^*(0) = (\frac{S_0}{K})^{\frac{1}{N}}$. The function is negative for all $u_\lambda > u^*(0)$, it attains a maximum in the interval $](1+r)^{T/N}, u^*[$. The root and the maximum decrease as K increases.

$$0 < j < \frac{N}{2}$$

- $C_0(j, K) = 0$ and $C_\lambda(j, 0) > 0$,
 $\lim_{u_\lambda \rightarrow +\infty} C_\lambda(j, K) = 0$
- If $K \geq S_0(1+r)^{\frac{(2j-N)T}{N}}$ then $C_\lambda(j, K)$ is negative for all u_λ .
- If $K < S_0(1+r)^{\frac{(2j-N)T}{N}}$ then $C_\lambda(j, K)$ has a root, $u^*(j) = (\frac{K}{S_0})^{\frac{1}{2j-N}}$. The function is negative for all $u_\lambda > u^*(j)$, it attains a maximum in the interval $](1+r)^{T/N}, u^*[$. The root and the maximum decrease as K increases.
- Since the function converges to zero it attains a minimum in the interval $]u^*(j), +\infty[$.

$$j = N/2, j \text{ is odd}$$

$$\text{Since } C_\lambda(j, K) = (S_0 - K) \binom{N}{j} (p_\lambda(1 - p_\lambda))^{N/2},$$

- $C_0(j, K) = 0$ and $\lim_{u_\lambda \rightarrow +\infty} C_\lambda(j, K) = 0$.
- The function is positive for all u_λ if $S_0 > K$ and negative for all u_λ if $S_0 < K$.
- The function attains a maximum for $u_\lambda = u^*$ if $S_0 > K$ and a minimum for $u_\lambda = u^*$ if $S_0 < K$.

$$\frac{N}{2} < j < N$$

- $C_0(j, K) = 0$ and $\lim_{u_\lambda \rightarrow +\infty} C_\lambda(j, K) = 0$
- If $(1+r)^{T/N} \leq \frac{N}{2\sqrt{(N-j)j}}$
 or
 $(1+r)^{T/N} > \frac{N}{2\sqrt{(N-j)j}}$ and $\frac{N}{2} < j \leq \frac{Nu^*}{2(1+r)^{T/N}}$
 then
 - If $K \leq S_0(1+r)^{\frac{(2j-N)T}{N}}$ the function is positive for all u_λ and attains a maximum, larger then u^* . The maximum increases as K increases.

- If $K > S_0(1+r)^{\frac{(2j-N)T}{N}}$ the function has a root, $u^*(j) = (\frac{K}{S_0})^{\frac{1}{2j-N}}$. The function is positive for all $u_\lambda > u^*(j)$. It attains a maximum, larger than u^* . The maximum and the root increase as K increases. It attains a minimum, smaller than $u_1^*(j)$, between $(1+r)^{T/N}$ and the root.
- $(1+r)^{T/N} > \frac{N}{2\sqrt{(N-j)j}}$ and $\frac{Nu^*}{2(1+r)^{T/N}} < j < N$
 - If $K \leq S_0(1+r)^{\frac{T(2j-N)}{N}}$ then the function is positive for all u_λ . It attains a maximum and a minimum in $]u_2(j), u^*[$. If K increases the difference between the maximum and the minimum becomes insignificant. It also attains a maximum which is larger than $u_1(j)$.
 - If $S_0(1+r)^{\frac{T(2j-N)}{N}} < K$ then the function has a root, $u^*(j) = (\frac{K}{S_0})^{\frac{1}{2j-N}}$, and is negative for all $u_\lambda < u^*(j)$. It attains a minimum, smaller than $u_1^*(j)$, between $(1+r)^{T/N}$ and the root.

$$\boxed{j = N}$$

- $C_0(N, K) = S_0(1+r)^T - K$ and $\lim_{u_\lambda \rightarrow \infty} C_\lambda(N, K) = S_0(1+r)^T$.
- If $K \geq S_0(1+r)^T$ then $C_\lambda(N, K)$ has a root, $u^*(N) = (\frac{K}{S_0})^{\frac{1}{N}}$. The function is positive for all $u_\lambda > u^*(j)$. The root decreases when K decreases. The function increases.
- If $S_0(1+r)^{T-\frac{2T}{N}} \leq K < S_0(1+r)^T$ then the function is positive and increasing for all u_λ .
- If $0 \leq K < S_0(1+r)^{T-\frac{2T}{N}}$ then the function is positive for all u_λ and attains a minimum in $](1+r)^{T/N}, u^*[$. The minimum decreases as K increases.

8 Procedure for the Pricing of the European Call Option

Suppose that K is such that $C_\lambda(j, K)$ is positive for all j , then the price $EC(K, T)$ (1) or (6) reads

$$\begin{aligned} C_\lambda(K) &= \frac{1}{(1+r)^T} \sum_{j=0}^N (S_0 u_\lambda^{2j-N} - K) \binom{N}{j} p_\lambda^j (1-p_\lambda)^{N-j} \\ &= \frac{S_0}{(1+r)^T} \sum_{j=0}^N (u_\lambda p_\lambda)^j (u_\lambda^{-1} (1-p_\lambda))^{N-j} \\ &\quad - \frac{K}{(1+r)^T} \sum_{j=0}^N p_\lambda^j (1-p_\lambda)^{N-j} \\ &= \frac{S_0}{(1+r)^T} (u_\lambda p_\lambda + u_\lambda^{-1} (1-p_\lambda))^N - \frac{K}{(1+r)^T} \\ &= S_0 - \frac{K}{(1+r)^T}, \end{aligned}$$

where in the last equality we applied (2).

This case is only possible if $K < S_0$, since otherwise the terms for $j < N/2$ are not in the sum. If this condition is fulfilled for K , then all terms for $j \geq N/2$ are in the sum. Therefore we concentrate on the terms with $j < N/2$. The expression $C_\lambda(j, K)$ is positive for all $u_\lambda < (\frac{S_0}{K})^{\frac{1}{N-2j}}$.

The smallest root is $(\frac{S_0}{K})^{\frac{1}{N}}$. This root is larger than $(1+r)^{T/N}$ if $0 < K \leq S_0(1+r)^{T/N}$. If, in this case,

$$(1+r)^{T/N} < u_\lambda < (\frac{S_0}{K})^{\frac{1}{N}}$$

then all terms are in the sum and $C_\lambda(K)$ is constant for those values of u_λ , namely $C_\lambda(K) = S_0(1+r)^{T/N} - K$.

If u_λ increases:

$$(\frac{S_0}{K})^{\frac{1}{N}} \leq u_\lambda < (\frac{S_0}{K})^{\frac{1}{N-2}}$$

then $C_\lambda(0, K) < 0$ and the corresponding term is not in the sum. Thus $C_\lambda(K) = S_0 - K(1+r)^{-T} - C_\lambda(0, K)(1+r)^{-T}$. Since $C_\lambda(0, K)$ is negative and decreasing for $(\frac{S_0}{K})^{\frac{1}{N}} \leq u_\lambda < (\frac{S_0}{K})^{\frac{1}{N-2}}$, $C_\lambda(K)$ increases for those values of u_λ .

This procedure can be extended for all values of u_λ and K .

Finally, we illustrate the procedure by an example in the case the imprecise volatility is described by a fuzzy quantity.

Let $0 < K \leq S_0(1+r)^{T/N}$ and

$$\begin{aligned} u_1 &\in \left] (1+r)^{T/N}, \left(\frac{S_0}{K}\right)^{\frac{1}{N}} \right[\\ u_2 &= \left(\frac{S_0}{K}\right)^{\frac{1}{N}} \\ u_3 &\in \left] \left(\frac{S_0}{K}\right)^{\frac{1}{N}}, \frac{S_0}{K} \right[. \end{aligned}$$

then, by applying (7), the α -cuts of the option price are

$$\left[S_0 - \frac{K}{(1+r)^T}, S_0 - \frac{K}{(1+r)^T} - \frac{C_{\lambda_{\alpha}^*(1)}(0, K)}{(1+r)^T} \right].$$

9 Conclusions

In the continuous Black & Scholes model, of which the binary tree model is a discrete time version, the price of a European call option is a strictly increasing function of the volatility, since the hedging parameter vega, i.e. the derivative of the price with respect to the volatility, is strictly positive.

In the discrete case we studied the functional behaviour of the price in order to model the uncertainty in the volatility. We can conclude that in the binary tree model the price is not necessarily a strictly increasing function of the volatility. As further research we will investigate the functional behaviour when this discrete time model converges to the Black & Scholes model.

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