Products of Capacities Derived from Additive Measures

Extended abstract

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Abstract

A new approach to define a product of capacities is presented. It works for capacities that are in a certain relation with additive measures, most often this means that they are somehow derived from additive measures. The product obtained is not unique, but rather, lower and upper bound are given.

Keywords

products of capacities, capacities, non-additive measures, increasing capacities, distorted measures

1 Introduction

It is a well known fact that there is no straightforward unique way to generalize the product of additive measures to the non-additive case. Several approaches to define a product for a specific family of non-additive measures, also called capacities, have already been proposed (see [3, 4, 6]). In this paper a new approach is presented to define a product for a family of capacities related to additive measures. The product of capacities defined here is in a close relation with the product of the corresponding additive measures.

Let us first explain the terminology used in this paper. Let S be a nonempty set and $\mathcal{A}$ a σ-algebra of its subsets. A *capacity* is a monotone function $v\colon \mathcal{A} \to \mathbb{R}$, such that $v(\emptyset) = 0$ and $v(S) < \infty$. Additive measures used here are assumed to be finite and defined on the same algebras as the capacities. We will also use the standard terminology for the products in additive case. So $\mu \times \lambda$ will be the usual additive product of two additive measures μ and λ, and $\mathcal{A} \times \mathcal{B}$ will be the usual product algebra.

A *product* of capacities u and v on σ-algebras $\mathcal{A}$ and $\mathcal{B}$ respectively, is any capacity $w\colon \mathcal{A} \times \mathcal{B} \to \mathbb{R}$ such that

$$w(A \times B) = u(A)v(B). \tag{1}$$

In additive measure theory the above condition uniquely determines the product of measures. Uniqueness crucially depends on additivity, moreover, without additivity requirement uniqueness of product can in general not be achieved. However, there always exist product capacities that satisfy (1), as shown in [4] where the lower and the upper bound are also given. But the set of all products if monotonicity of the product alone is required is far too big and their values differ too much on non-rectangular sets.

In order to reduce the set of all possible product capacities, the products are sought within some class of capacities with some additional properties that are preserved by multiplication. In [4] Hendon et al. define a product of *belief functions* using the idea of *Möbius representation* of capacities. Another definition of a product was proposed by Koshevoy in [6] using triangulation of geometrical realizations of distributive lattices. Denneberg in [3] joins both ideas to obtain a definition of a product for general monotone capacities which coincides with the Möbius product for the class of belief functions.

Instead of restricting to a special class of capacities Ghirardato in [5] restricts to a special class of functions for which the Fubini theorem for capacities holds. This class contains characteristic functions for a family of sets that he calls *comonotonic sets*. For these sets the double integral of their characteristic functions is a natural definition of a product of the capacities.

Although the existing definitions of products cover a very general class of capacities, most of them are still limited to discrete capacities. In this paper I present a definition of a product of capacities that seems to work better for continuous capacities, however, the results are valid for discrete case as well. The class of capacities it covers is rather restricted, but I think there are ways open to generalize this idea.

2 Increasing Capacities

The product of capacities defined here works for a family of capacities that are in a certain way related to additive measures. Before defining this relation, we will observe it in the case of a supermodular distorted measure. A capacity v is a *distorted measure* if it can be expressed as a composite $f \circ \mu$, where μ is an additive measure and the distortion f is an increasing real function with $f(0) = 0$. It is well known that a distorted measure is submodular or supermodular if the distortion is concave or convex respectively (see [2]). Suppose now that v is a supermodular distorted measure with distortion f applied to measure μ. Since f is a convex function, graph of a linear function intersects its graph in at most two different points. Using this fact, one can easily observe that for each pair of subsets $A \subseteq B$, $v(A)/\mu(A) \leq v(B)/\mu(B)$ holds. This leads to the next definition.

Definition 1 *Let μ be an additive measure on a σ-algebra $\mathcal{A}$ and v a capacity on the same algebra. The capacity v is* increasing *with respect to μ if the following* *is true: If $A \subseteq B$ and $\mu(A) > 0$ then $v(A)/\mu(A) \leq v(B)/\mu(B)$ and if $\mu(A) = 0$ then also $v(A) = 0$.*

If $\mu\colon \mathcal{A} \to \mathbb{R}$ is an additive measure, $I(\mu)$ will denote the set of all increasing capacities with respect to μ.

Further, we define quotient $m_v(A) := v(A)/\mu(A)$, where $m_v(A) = 0$ for all A with $v(A) = \mu(A) = 0$, and for each $t \in \mathbb{R}$, $\mathcal{A}_{v,t} := \left\{ A \middle|\, t \leq \frac{v(A)}{\mu(A)} \right\}$. According to Definition 1, $m_v\colon \mathcal{A} \to \mathbb{R}$ is an increasing set function and it will be used to define the product of capacities. Thus, the product of two increasing capacities u and v will be defined by defining the corresponding $m_{u\times v}$.

We will also generalize the concept of comonotonicity for the case of increasing capacities. (For definition of comonotonicity for real functions see e.g. [2]). If v_1 and v_2 are capacities on a σ-algebra $\mathcal{A}$, increasing with respect to an additive measure μ, then we say v_1 and v_2 are *comonotonic* if the union $\{\mathcal{A}_{v_1,t} | t \in \mathbb{R}\}$ $\cup \{\mathcal{A}_{v_2,s} | s \in \mathbb{R}\}$ forms a chain of subsets of $\mathcal{A}$. Equivalently, capacities v_1 and v_2 are comonotonic exactly when m_{v_1} and m_{v_2} are comonotonic as real functions on $\mathcal{A}$ in the usual sense.

3 Products of Increasing Capacities

Given a set $C \in \mathcal{A} \times \mathcal{B}$, we will first define two Borel measurable sets in $\mathbb{R}^2$ whose Lebesgue measures are the minimum and the maximum value for the function $m_{u\times v}$. These sets can be considered as some kind of products of m_u and m_v.

Definition 2 *Let u and v be increasing capacities with respect to measures μ and λ respectively and defined on σ-algebras $\mathcal{A}$ and $\mathcal{B}$. Let $\mathcal{A} \times \mathcal{B}$ be the algebra of all measurable sets with respect to the product measure $\mu \times \lambda$. Define functions $\underline{\varphi}_{u,v}$ and $\overline{\varphi}_{u,v}\colon \mathcal{A} \times \mathcal{B} \to 2^{\mathbb{R}^2}$ with*

$$
\begin{aligned}
(x,y) \in \underline{\varphi}_{u,v}(C) \quad &\iff \quad \textit{If there exist } A \in \mathcal{A}_{u,x} \textit{ and } B \in \mathcal{B}_{v,y} \\
&\qquad \textit{such that } A \times B \subseteq C, x > 0, y > 0 \\
(x,y) \in \overline{\varphi}_{u,v}(C) \quad &\iff \quad \textit{If for all } A \in \mathcal{A} \textit{ and } B \in \mathcal{B} \textit{ such that } A \times B \supseteq C \\
&\qquad A \in \mathcal{A}_{u,x} \textit{ and } B \in \mathcal{B}_{v,y} \textit{ holds, } x > 0, y > 0
\end{aligned}
$$

It is easy to see that $\underline{\varphi}_{u,v}(C)$ and $\overline{\varphi}_{u,v}(C)$ are Borel measurable sets in $\mathbb{R}^2$ for all $C \in \mathcal{A} \times \mathcal{B}$. However, there is a substantial asymmetry between both sets. While $\overline{\varphi}_{u,v}(C)$ is only a rectangle that represents the smallest rectangular set (with respect to m_u and m_v) that contains C, $\underline{\varphi}_{u,v}(C)$ is a union of rectangles representing the family of the largest rectangular sets that are contained in C. Clearly, the latter set therefore characterizes C much more precisely, in general, than the former one.

The definition of the lower and the upper bound for a product follows.

Definition 3 *Let u and v be increasing capacities with respect to measures μ and λ. We define the* lower product *of u and v as*

$$(\underline{u \times v})(C) = \mu_{\mathbb{R}^2}\left(\underline{\varphi}_{u,v}(C)\right)(\mu \times \lambda)(C)$$

and their upper product *as*

$$(\overline{u \times v})(C) = \mu_{\mathbb{R}^2}\left(\overline{\varphi}_{u,v}(C)\right)(\mu \times \lambda)(C).$$

The products $\underline{u \times v}$ and $\overline{u \times v}$ turn out to be the lower and the upper bound for a product of capacities under some additional natural assumptions. But first we state some properties of the products just defined.

Proposition 1 *The following statements hold for $u, u', u_i \in I(\mu)$ and $v \in I(\lambda)$.*

(i) If $u \leq u'$ then $\underline{u \times v} \leq \underline{u' \times v}$.

(ii) $\underline{(u+u') \times v} \leq \underline{u \times v} + \underline{u' \times v}$, equality holds if u and u' are comonotonic.

(iii) If $u_i \nearrow u$ then $\underline{u_i \times v} \nearrow \underline{u \times v}$.

and

(i)' If $u \leq u'$ then $\overline{u \times v} \leq \overline{u' \times v}$.

(ii)' $\overline{(u+u') \times v} = \overline{u \times v} + \overline{u' \times v}$

(iii)' If $u_i \nearrow u$ then $\overline{u_i \times v} \nearrow \overline{u \times v}$.

Because of symmetry of the product all of the above properties also hold for the second term.

The above properties also show that the upper and the lower product are not symmetric, as one might expect. While the upper product is additive, the lower is only comonotonically additive.

In order to prove that the lower and the upper product are indeed lower and upper bound in a family of product operators, we define operators $\underline{\Phi}$ and $\overline{\Phi}$: $I(\mu) \times I(\lambda) \to I(\mu \times \lambda)$ with $\underline{\Phi}(u,v) = \underline{u \times v}$ and $\overline{\Phi}(u,v) = \overline{u \times v}$.

Proposition 1 implies that the operators $\underline{\Phi}$ and $\overline{\Phi}$ are monotonic and continuous from below (in the sense of [2]) in both terms. The upper product operator $\overline{\Phi}$ is also biadditive, while the lower product operator $\underline{\Phi}$ is subadditive in both terms, however, when applied to sum of comonotonic capacities it is additive as well. Usually such operators are said to be *comonotonically additive*.

The following two theorems are the main results of this paper.

Theorem 1 *Let μ and λ be positive measures on σ-algebras $\mathcal{A}$ and $\mathcal{B}$ respectively. Let $\Phi\colon\ I(\mu)\times I(\lambda)\to I(\mu\times\lambda)$ be an operator that is comonotonically additive, positively homogeneous and continuous in both terms and such that*

$$\Phi(u,v)(A\times B)=u(A)v(B),$$

for all $A\in\mathcal{A}$ and $B\in\mathcal{B}$. Then $\underline{\Phi}\le\Phi\le\overline{\Phi}$ holds.

Proof Sketch. To prove this and also the next theorem, we define a family of simple increasing capacities that we call *cut measures*. Let $\mathcal{A}'\subseteq\mathcal{A}$ be a family of sets such that for each pair of sets $A\subseteq B$, $A\in\mathcal{A}'$ implies $B\in\mathcal{A}'$. Then we define cut measure $\mu|_{\mathcal{A}'}$ by

$$\mu|_{\mathcal{A}'}(A):=\begin{cases}\mu(A) & \text{if } A\in\mathcal{A}'\\ 0 & \text{otherwise}\end{cases}$$

The first step of proof is to verify that $\underline{u\times v}$ and $\overline{u\times v}$ are the smallest and the greatest product measures in case where u and v are cut measures, say $u=\mu|_{\mathcal{A}'}$ and $v=\lambda|_{\mathcal{B}'}$. It turns out that cut measures $(\mu\times\lambda)|_{\mathcal{C}'}$ and $(\mu\times\lambda)|_{\mathcal{C}''}$ are their smallest and largest product capacities increasing with respect to $\mu\times\lambda$, where $\mathcal{C}'=\{C|\text{ there exist } A\in\mathcal{A} \text{ and } B\in\mathcal{B} \text{ such that } A\times B\subseteq C\}$ and $\mathcal{C}''=\{C|\text{ if for all } A\times B\supseteq C,\ A\in\mathcal{A} \text{ and } B\in\mathcal{B} \text{ holds}\}$. These two cut measures turn out to be equal to $\underline{u\times v}$ and $\overline{u\times v}$ respectively.

The second step is to show that an increasing capacity can be uniformly approximated by sums of comonotonic cut measures. Using first step, comonotonic additivity and continuity of Φ we get the desired inequality. □

Next important property that a product should have is associativity.

Theorem 2 *Let u,v and w be increasing capacities, with respect to μ,λ and η. Then the following equalities hold:*

$$\underline{\underline{u\times v}\times w}=\underline{u\times\underline{v\times w}}=:\underline{u\times v\times w}$$

and

$$\overline{\overline{u\times v}\times w}=\overline{u\times\overline{v\times w}}=:\overline{u\times v\times w}.$$

The proof of this theorem also consists of two steps, the first being proof that it holds for the case of cut measures and the second one is extension to general case, using comonotonic additivity and continuity of Φ.

4 Conclusion

The results presented here, should be extended to more general families of capacities. One idea is to extend the product to differences of increasing measures.

That is, if capacities u and v can be written as $u = u_1 - u_2$ and $v = v_1 - v_2$, where u_1, u_2, v_1, v_2 are increasing with respect to some additive measure μ and λ respectively, an obvious way to extend present definition of the product would be, to define the lower product $\underline{u \times v} = \underline{u_1 \times v_1} + \underline{u_2 \times v_2} - \underline{u_1 \times v_2} - \underline{u_2 \times v_1}$. Such a definition unfortunately does not provide uniqueness of the product. A topic of further study is therefore searching for alternative generalizations.

The main disadvantage of the product defined here is, that it depends on the underlying additive measure. If we, on the other hand, modified the definition to allow all additive measures and apply minimum or maximum on it, we would probably obtain a trivial result. A compromise would be, to consider a proper family of additive measures. Such a family could depend on the type of considered capacities.

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