A Second-Order Uncertainty Model of Independent Random Variables: An Example of the Stress-Strength Reliability

L.V. UTKIN
University of Munich, Germany
St.Petersburg State Forest Technical Academy, Russia

Abstract

A second-order hierarchical uncertainty model of a system of independent random variables is studied in the paper. It is shown that the complex non-linear optimization problem for reducing the second-order model to the first-order one can be represented as a finite set of simple linear programming problems with a finite number of constraints. The stress-strength reliability analysis by unreliable information about statistical parameters of the stress and strength exemplifies the model. Numerical examples illustrate the proposed algorithm for computing the stress-strength reliability.

Keywords

stress-strength reliability, imprecise probabilities, second-order uncertainty, natural extension, previsions, linear programming

1 Introduction

By processing unreliable information, much attention have been focused on the *second-order uncertainty models* (*hierarchical uncertainty models*) due to their quite commonality. These models describe the uncertainty of a random quantity by means of two levels. Various second-order models and their applications can be found in the literature [4, 6, 7, 12, 13, 22], and a comprehensive review of hierarchical models is given in [5], where it is argued that the most common hierarchical model is the Bayesian one [2, 8, 14]. At the same time, the Bayesian hierarchical model is unrealistic in problems where there is available only partial information about the system behavior.

The main shortcoming of most proposed second-order hierarchical models (from the informational point of view) is the necessity to assume the certain type of the second-order probability or possibility distributions defined on the first-order level. This information is usually absent in many applications and additional assumptions may lead to some inaccuracy in results. The study of some tasks related to homogeneous second-order models without any assumptions about probability distributions has been illustrated by Kozine and Utkin [10]. However, these models are of limited use due to the homogeneity of gambles considered on the first-order level, i.e., the initial information is restricted by previsions of identical gambles. A new hierarchical uncertainty model for combining different types of evidence was proposed by Utkin [17, 16], where the second-order probabilities can be regarded as confidence weights and the first-order uncertainty is modelled by lower and upper previsions of different gambles [21]. However, the proposed model [17, 16] supposes that initial information is given only for one random variable. At the same time, many applications use a set of random variables described by a second-order uncertainty model, and it is necessary to find a model for some function of these variables. For example, reliability analysis demands to compute the reliability of a system under uncertain information about its components. An imprecise hierarchical model of a number of random variables has been studied by Utkin [18], but this model supposes that there is no information about independence of random variables. It should be noted that the condition of independence takes place in many applications. This condition makes the natural extension to be non-linear and, as a result, the corresponding hierarchical model becomes very complex.

An efficient approach to solve this problem is proposed in the paper. In order to show the practical relevance of the proposed approach, it is applied to the stress-strength reliability analysis by the independent stress and strength.

2 Imprecise Stress-Strength Reliability

A probabilistic model of structural reliability can be formulated as follows. Let Y represent a random variable describing the strength of a system and let X represent a random variable describing the stress or load placed on the system. System failure occurs when the stress on the system exceeds the strength of the system. Then the reliability of the system is determined as $R = \Pr\{X \leq Y\}$. A general approach to the structural reliability analysis based on the imprecise probability theory [11, 21, 23] was proposed in [19, 20]. Let us briefly consider this approach. Suppose that available information about the random stress X and the random strength Y is given as a set of n lower $\underline{\mathbb{E}}h_i$ and upper $\overline{\mathbb{E}}h_i$ previsions of gambles $h_i(X,Y)$ (unbounded gambles are considered in [15]) such that

$$\underline{\mathbb{E}}h_i \leq \mathbb{E}_{p(x,y)}h_i(X,Y) \leq \overline{\mathbb{E}}h_i,\ i = 1,...,n.$$

Here $p(x,y)$ is a joint density of the stress and strength. It is assumed that there exist a set of density functions such that linear previsions $\mathbb{E}_{p(x,y)}h_i$ can be regarded as expectations of h_i. Taking into account that

$$R = \Pr\{X \leq Y\} = \mathbb{E}_{p(x,y)}I_{[0,\infty)}(Y - X), \tag{1}$$

we can write the following optimization problems (natural extension) for computing the lower $\underline{R}$ and upper $\overline{R}$ stress-strength reliability as follows:

$$\underline{R}\left\langle\overline{R}\right\rangle=\inf_{p}\left\langle\sup_{p}\right\rangle\int_{\mathbb{R}_{+}^{2}}I_{[0,\infty)}(y-x)p(x,y)\mathrm{d}x\mathrm{d}y, \quad (2)$$

subject to

$$\underline{\mathbb{E}}h_i\leq\int_{\mathbb{R}_{+}^{2}}h_i(x,y)p(x,y)\mathrm{d}x\mathrm{d}y\leq\overline{\mathbb{E}}h_i,\ i=1,...,n. \quad (3)$$

Here the infimum and supremum are taken over the set of all possible densities $\{p(x,y)\}$ satisfying conditions (3), $I_{[0,\infty)}(Y-X)$ is the indicator function taking the value 1 if $Y\geq X$ and 0 otherwise. If random variables X and Y are independent, then the constraint $p(x,y)=p_X(x)p_Y(y)$ is added to constraints (3), where p_X and p_Y are densities of X and Y, respectively.

The natural extension is a powerful tool for analyzing the reliability on the basis of available partial information. However, it has a shortcoming. Let us imagine that two experts provide the following judgements about the stress: (i) mean value of the stress is not greater than 10; (ii) mean value of the stress is not less than 10 hours. The natural extension produces the resulting mean value $[0,10]\cap[10,\infty)=10$. In other words, the absolutely precise measure is obtained from too imprecise initial data. This is unrealistic in practice of reliability analysis. The reason of such results is that probabilities of judgements are assumed to be 1. If we assign some different probabilities to judgements, then we obtain more realistic assessments. For example, if the belief to each judgement is 0.5, then, according to [9], the resulting mean value is greater than 5 hours. Therefore, in order to obtain the accurate and realistic reliability assessments, it is necessary to take into account some vagueness of information about characteristics of the stress and strength.

3 Second-Order Model. Problem Statement

Suppose that there exist n judgements about the stress X:

$$\mathbb{E}f_j(X)\in T_j=[\underline{t}_j,\overline{t}_j],\ j=1,...,n,$$

and l judgements about the strength Y:

$$\mathbb{E}h_j(Y)\in S_j=[\underline{s}_j,\overline{s}_j],\ j=1,...,l.$$

Here f_j and h_j are gambles corresponding to the available judgements about X and Y. Moreover, it is known that

$$\underline{\alpha}_j\leq\Pr\left\{\mathbb{E}f_j\in T_j\right\}\leq\overline{\alpha}_j,\ j=1,...,n,$$
$$\underline{\beta}_j\leq\Pr\left\{\mathbb{E}h_j\in S_j\right\}\leq\overline{\beta}_j,\ j=1,...,l.$$

The second-order probabilities $[\underline{\alpha}_j, \overline{\alpha}_j]$ and $[\underline{\beta}_j, \overline{\beta}_j]$ are interpreted as a model for uncertainty about "correct" values of partially known measures of X and Y. Let us briefly discuss the sense of beliefs to the expert judgements. If we know that an expert provides $100 \cdot \alpha\%$ of "correct" judgements, this means that, by giving finitely many intervals, say n, for an unknown parameter, approximately $n\alpha$ intervals cover some "correct" value of the parameter. But if we have only the $(n+1)$-st interval and do not know anything about previous n intervals, then we can only say that the "correct" value of the parameter lies in this interval with probability α and outside this interval with probability $1-\alpha$. If we would have all aforementioned n intervals, some probability distribution of the parameter could be constructed and well-known Bayesian methods could be used. In this case, there is no need to apply imprecise probabilities.

The term "expert information" may be used in a more general sense. In particular, confidence intervals of parameters elicited as a result of statistical inference with corresponding confidence probabilities may be regarded as "beliefs to experts". For example, if we have one confidence interval for the expectation of a probability distribution, then we can only assert, that the "correct" value of the expectation is in the interval with the confidence interval probability $[\alpha, 1]$ and outside the confidence interval with the probability $[0, 1-\alpha]$.

How to find average values of $\underline{R}$ and $\overline{R}$, i.e., to reduce the second-order model to the first-order one? Roughly speaking, if we have second-order probabilities defined for different intervals of $\mathbb{E}f_j$ and $\mathbb{E}h_j$, then there exist a set of second-order distributions of $\mathbb{E}f_j$, $\mathbb{E}h_j$, and $\mathbb{E}I_{[0,\infty)}(Y-X)$ produced an interval of lower and upper expectations of $\mathbb{E}I_{[0,\infty)}(Y-X)$, i.e., $\underline{R}$ and $\overline{R}$. We will call this interval "average" to distinguish expectations (previsions) on the first and second levels of the considered second-order uncertainty model. In fact, the "average" interval allows us to get rid of the more complex second-order model and to deal with the first-order model. This problem is especially difficult if the stress and strength are independent. At that, a special type of independence called by the free product [11] is studied in the paper. This type of independence is like to the epistemic irrelevance [3] and, generally, is asymmetric.

In order to give the reader the essence of the subject analyzed and make all the formulas more readable, we will mainly consider only the lower bound $\underline{R}$.

Let $v_i = \mathbb{E}f_i$ and $w_i = \mathbb{E}h_i$ be values of random variables V_i and W_i defined on sample spaces Ω_i and Λ_i, respectively. Let $V = (V_1, ..., V_n)$, $W = (W_1, ..., W_n)$ and $\mathbf{V} = (v_1, ..., v_n)$, $\mathbf{W} = (w_1, ..., w_l)$ be the vectors of random variables V_i, W_i and their values, respectively. Denote $N = \{1, ..., n\}$ and $L = \{1, ..., l\}$. Then the natural extension for computing $\underline{R}$ can be written as a sequence of lower expectations:

$$\underline{R} = \underline{\mathbb{E}}^{\mathbf{W}} \left\{ \underline{\mathbb{E}}^{\mathbf{V}|\mathbf{W}} \left(\mathbb{E}I_{[0,\infty)}(Y-X) \right) \right\}$$

by given lower and upper previsions

$$\underline{\mathbb{E}}I_{T_i}(v_i) = \underline{\alpha}_i,\ \overline{\mathbb{E}}I_{T_i}(v_i) = \overline{\alpha}_i,\ \underline{\mathbb{E}}I_{S_i}(w_i) = \underline{\beta}_i,\ \overline{\mathbb{E}}I_{S_i}(w_i) = \overline{\beta}_i. \tag{4}$$

By introducing a random variable Z having values $z(\mathbf{V},\mathbf{W}) = \mathbb{E}I_{[0,\infty)}(Y-X)$ and assuming that there exists a set of densities $\varphi(\mathbf{V})$ and $\psi(\mathbf{W})$ of vectors V and W, respectively, we can write

$$\underline{R} = \inf_{\psi} \int_{\Lambda} \left(\inf_{\varphi} \int_{\Omega} z(\mathbf{V},\mathbf{W})\varphi(\mathbf{V})\mathrm{d}\mathbf{V} \right) \psi(\mathbf{W})\mathrm{d}\mathbf{W}, \tag{5}$$

subject to

$$\underline{\alpha}_i \leq \int_{\Omega} I_{T_i}(v_i)\varphi(\mathbf{V})\mathrm{d}\mathbf{V} \leq \overline{\alpha}_i,\ i \in N,\ \underline{\beta}_i \leq \int_{\Lambda} I_{S_i}(w_i)\psi(\mathbf{W})\mathrm{d}\mathbf{W} \leq \overline{\beta}_i,\ i \in L. \tag{6}$$

Here $\Omega = \Omega_1 \times \cdots \times \Omega_n$, $\Lambda = \Lambda_1 \times \cdots \times \Lambda_l$. The sample spaces Ω_i and Λ_j are determined by sets of values $\mathbb{E}f_i$ and $\mathbb{E}h_j$, i.e.,

$$\Omega_i = [\inf \mathbb{E}f_i, \sup \mathbb{E}f_i],\ \Lambda_j = [\inf \mathbb{E}h_j, \sup \mathbb{E}h_j].$$

A dual optimization problem can not be written as it has been made in [18] because the initial problem is non-linear. Our aim is to find $\underline{R}$, i.e., to solve (5)-(6).

4 Solution of Problem (5)-(6)

4.1 A Set of Linear Programming Problems

Let $\mathbf{W}^* = (w_1^*,...,w_n^*) \in \Lambda$ be a realization of the vector $\mathbf{W}$. Denote $R(\mathbf{W}^*) = \mathbb{E}_{\varphi} z(\mathbf{V},\mathbf{W}^*)$. Problem (5)-(6) can be represented as follows:

$$\begin{aligned} \underline{R} &= \inf_{\psi} \int_{\Lambda} \left(\inf_{\varphi} \int_{\Omega} z(\mathbf{V},\mathbf{W})\varphi(\mathbf{V})\mathrm{d}\mathbf{V} \right) \psi(\mathbf{W})\mathrm{d}\mathbf{W} \\ &= \inf_{\psi} \int_{\Omega} \inf_{\varphi} R(\mathbf{W}^*)\psi(\mathbf{W})\mathrm{d}\mathbf{W} = \inf_{\psi} \int_{\Omega} \underline{R}(\mathbf{W}^*)\psi(\mathbf{W})\mathrm{d}\mathbf{W}, \end{aligned} \tag{7}$$

subject to

$$\underline{\beta}_i \leq \mathbb{E}_{\psi} I_{S_i}(w_i) \leq \overline{\beta}_i,\ i \in L. \tag{8}$$

Here

$$\underline{R}(\mathbf{W}^*) = \inf_{\varphi} \mathbb{E}_{\varphi} z(\mathbf{V},\mathbf{W}^*), \tag{9}$$

subject to

$$\underline{\alpha}_i \leq \mathbb{E}_{\varphi} I_{T_i}(v_i) \leq \overline{\alpha}_i,\ i = 1,...,n. \tag{10}$$

Problems (7)-(8) and (9)-(10) are linear and dual optimization problems can be written, i.e., we have a set of the following problems for each $\mathbf{W}^* \in \Lambda$:

$$\underline{R}(\mathbf{W}^*) = \sup \left(c_0 + \sum_{i \in N} (c_i \underline{\alpha}_i - d_i \overline{\alpha}_i) \right), \tag{11}$$

subject to $c_i, d_i \in \mathbb{R}_+$, $c_0 \in \mathbb{R}$, $i \in N$, and $\forall \mathbf{V} \in \Omega$,

$$c_0 + \sum_{i \in N} (c_i - d_i) I_{T_i}(v_i) \leq z(\mathbf{V}, \mathbf{W}^*), \tag{12}$$

and one linear programming problem

$$\underline{R} = \sup \left(c_0 + \sum_{i \in L} \left(c_i \underline{\beta}_i - d_i \overline{\beta}_i \right) \right), \tag{13}$$

subject to $c_i, d_i \in \mathbb{R}_+$, $c_0 \in \mathbb{R}$, $i \in L$, and $\forall \mathbf{W}^* \in \Lambda$,

$$c_0 + \sum_{i \in L} (c_i - d_i) I_{S_i}(w_i) \leq \underline{R}(\mathbf{W}^*). \tag{14}$$

The dual problems have been introduced in order to get rid of densities $\varphi(\mathbf{V})$ and $\psi(\mathbf{W})$.

4.2 Solution of Problem (11)-(12)

An algorithm and an approach to solving a problem similar to (11)-(12) are given in [16, 17]. But problem (11)-(12) has some difference. To solve this problem, it is necessary to define what $z(\mathbf{V}, \mathbf{W}^*)$ is.

Let J be a set of indices and $J \subseteq N$. Introduce the following sets of constraints:

$$\mathcal{T}_J = \{T_i,\ i \in J\},\ \mathcal{T}_J^c = \{T_i^c,\ i \in J\},\ T_i^c = \Omega_i \backslash T_i.$$

Then constraints (12) can be rewritten as

$$c_0 + \sum_{i=1}^{n} (c_i - d_i) I_{T_i}(\mathbb{E}_{p_X} f_i) \leq z(\mathbf{V}, \mathbf{W}^*),\ p_X \in \mathcal{P}. \tag{15}$$

Here $\mathcal{P}$ is the set of all densities $\{p_X\}$. Let us consider these constraints in detail and define $z(\mathbf{V}, \mathbf{W}^*)$. Note that

$$z(\mathbf{V}, \mathbf{W}^*) = \mathbb{E}_{p_X p_Y} I_{[0,\infty)}(Y - X). \tag{16}$$

However, we fixed the vector $\mathbf{W}^* = (\mathbb{E}^* h_1, ..., \mathbb{E}^* h_l)$. This means that the set of probability densities $p_Y(y)$ is restricted as follows:

$$\mathbb{E}^*_{p_Y} h_1 = w_1^*, ..., \mathbb{E}^*_{p_Y} h_l = w_l^*. \tag{17}$$

So, $z(\mathbf{V}, \mathbf{W}^*)$ can be found by solving the optimization problem with objective function (16), constraints (17), and constraints for p_X, which will be considered below.

In order to compute the indicator functions in (15), it is necessary to substitute different functions p_X from $\mathcal{P}$ and to calculate the corresponding values of $\mathbb{E}_{p_X} f_i$

and $I_{T_i}(\mathbb{E}_{p_X} f_i)$. Moreover, it is necessary to solve problem (16)-(17) for each $p_X \in \mathcal{P}$. Obviously, this task can not be practically solved. Therefore, another way for solving the optimization problem is proposed.

We call the set $\mathcal{T}^c_{N\setminus J} \cup \mathcal{T}_J$ *consistent* if there is at least one density p_X such that $\mathbb{E}_{p_X} f_i \in T_i$, $i \in J$, $\mathbb{E}_{p_X} f_j \in T^c_j$, $j \in N\setminus J$. Now we can see that if the set $\mathcal{T}_J \cup \mathcal{T}^c_{N\setminus J}$ is consistent, then $I_{T_i}(\mathbb{E}_{p_X} f_i) = 1$ if $i \in J$, and $I_{T_i}(\mathbb{E}_{p_X} f_i) = 0$ if $i \in N\setminus J$. In other words, if the set $\mathcal{T}^c_{N\setminus J} \cup \mathcal{T}_J$ is consistent, then there exists at least one density p_X such that all linear previsions $\mathbb{E}_{p_X} f_i$, $i \in J$, are in intervals T_i and their indicator functions are equal to 1, all linear previsions $\mathbb{E}_{p_X} f_j$, $j \in N\setminus J$, do not belong to intervals T_i and their indicator functions are equal to 0. In this case, we will say that p_X belongs to a set $\mathcal{P}_J$. So, to simplify constraints (15), it is necessary to look over all consistent sets $\mathcal{T}^c_{N\setminus J} \cup \mathcal{T}_J$. Then constraints (15) can be rewritten for all $J \subseteq N$, such that $\mathcal{T}^c_{N\setminus J} \cup \mathcal{T}_J$ are consistent, as follows:

$$c_0 + \sum_{i\in J} (c_i - d_i) \leq z(\mathbf{V}, \mathbf{W}^*). \tag{18}$$

If $\mathcal{T}^c_{N\setminus J} \cup \mathcal{T}_J$ is inconsistent, then corresponding inequality (18) is excluded from the list of all constraints.

But how to determine the consistency of sets $\mathcal{T}^c_{N\setminus J} \cup \mathcal{T}_J$? The set $\mathcal{T}^c_{N\setminus J} \cup \mathcal{T}_J$ is consistent if an optimization problem with constraints produced by $\mathcal{T}^c_{N\setminus J} \cup \mathcal{T}_J$ has any solution. At that, the objective function may be arbitrary. In other words, for determining the consistency of $\mathcal{T}^c_{N\setminus J} \cup \mathcal{T}_J$, it is necessary to solve the following optimization problem:

$$\inf_{p_X} \left(\sup_{p_X} \right) \mathbb{E}_{p_X} u(x),$$

subject to $\mathbb{E}_{p_X} f_i \in T_i$, $i \in J$, $\mathbb{E}_{p_X} f_j \in T^c_j$, $j \in N\setminus J$. Here u is an arbitrary function.

Let $p^{(1)}_X \in \mathcal{P}_J$ and $p^{(2)}_X \in \mathcal{P}_J$. Then

$$I_{T_i}(\mathbb{E}_{p^{(1)}_X} f_i) = I_{T_i}(\mathbb{E}_{p^{(2)}_X} f_i).$$

Let

$$z^{(2)}(\mathbf{V}, \mathbf{W}^*) = \mathbb{E}_{p^{(2)}_X p_Y} I_{[0,\infty)}(Y - X) \leq \mathbb{E}_{p^{(1)}_X p_Y} I_{[0,\infty)}(Y - X) = z^{(1)}(\mathbf{V}, \mathbf{W}^*).$$

Then the constraint

$$c_0 + \sum_{i\in N} (c_i - d_i) I_{T_i}(\mathbb{E}_{p^{(1)}_X} f_i) \leq z^{(1)}(\mathbf{V}, \mathbf{W}^*)$$

follows from the constraint

$$c_0 + \sum_{i\in N} (c_i - d_i) I_{T_i}(\mathbb{E}_{p^{(2)}_X} f_i) \leq z^{(2)}(\mathbf{V}, \mathbf{W}^*)$$

and can be removed. This implies that (15) is equivalent to

$$c_0 + \sum_{i \in J} (c_i - d_i) \leq \inf_{\mathcal{P}_J} z(\mathbf{V}, \mathbf{W}^*), \tag{19}$$

where

$$\inf_{\mathcal{P}_J} z(\mathbf{V}, \mathbf{W}^*) = \inf_{p_X, p_Y} \mathbb{E}_{p_X p_Y} I_{[0,\infty)}(Y - X), \tag{20}$$

subject to

$$\mathbb{E}_{p_X} f_i \in \begin{cases} T_i, & i \in J \\ T_i^c, & i \in N \backslash J \end{cases}, \ i \in N, \tag{21}$$

$$\mathbb{E}_{p_Y} h_i = w_i^*, \ i \in L. \tag{22}$$

So, an infinite number of constraints has been reduced to at most 2^n constraints (19). Since the function u is arbitrary, then $\inf_{\mathcal{P}_J} z(\mathbf{V}, \mathbf{W}^*)$ may be used in place of u. There exist exact analytical solutions to problem (20)-(22) for various types of initial information [19].

4.3 Solution of Problem (13)-(14)

Now we have the values of $\underline{R}(\mathbf{W}^*)$ for each $\mathbf{W}^* \in \Lambda$. Let us introduce the sets

$$\mathcal{S}_K = \{S_i, \ i \in K\}, \ \mathcal{S}_K^c = \{S_i^c, \ i \in K\}, \ K \subseteq L = \{1, 2, ..., l\}.$$

For solving problem (13)-(14), we apply an algorithm which is similar to the considered one in the previous subsection, i.e.,

$$\underline{R} = \sup \left(c_0 + \sum_{i \in L} \left(c_i \underline{\beta}_i - d_i \overline{\beta}_i \right) \right), \tag{23}$$

subject to $c_i, d_i \in \mathbb{R}_+$, $c_0 \in \mathbb{R}$, $i \in L$, and $\forall K \subseteq L$, $\forall \mathbf{W}^* \in \Lambda$,

$$c_0 + \sum_{i \in K} (c_i - d_i) \leq \inf_{\mathbf{W}^* \in \mathcal{S}_{L \backslash K}^c \cup \mathcal{S}_K} \underline{R}(\mathbf{W}^*). \tag{24}$$

This is a simple linear programming problem with at most 2^l constraints.

5 Exact Bounds for the Reliability

It can be seen from results of the previous section that complex non-linear optimization problem (5)-(6) is reduced to a set of linear programming problems with finitely many constraints and non-linear problems (20)-(22) which can be numerically solved or have exact solutions [19] for the most important types of initial information (points of probability distribution functions of X and Y, moments of X and Y, probabilities defined on nested intervals). However, these optimization problems have to be solved for all values of $\mathbf{W}^* \in \Lambda$ whose number may be infinite. This leads only to the approximate solution and makes the task to be rather difficult from the computational point of view even by a small number of initial judgements. It turns out that optimization problem (5)-(6) can be exactly solved. Therefore, an interesting and efficient solution of the problem is proposed in this section.

Let us consider constraints (24). Suppose that $\underline{R}(\mathbf{W}^*)$ achieves its minimum at $\mathbf{W}^* = \mathbf{W}_o^*(K) \in \mathcal{S}_{L\backslash K}^c \cup \mathcal{S}_K$. Then all vectors $\mathbf{W}^* \in \mathcal{S}_{L\backslash K}^c \cup \mathcal{S}_K$ except $\mathbf{W}_o^*(K)$ are not used in constraints to problem (23)-(24). This implies that we do not need to look over all possible vectors $\mathbf{W}^*$. By returning to problem (11)-(12), it is necessary to solve it only for $\mathbf{W}_o^*(K)$, $K \subseteq L$. This implies that the number of solved optimization problems is finite and depends on numbers n and l of initial judgements about X and Y. Moreover, we can obtain exact bounds for the stress-strength reliability in this case. However, we do not know points $\mathbf{W}_o^*(K)$ before solving problem (11)-(12). Let us show how to overcome this difficulty.

It follows from (11)-(12) that $\underline{R}(\mathbf{W}^*)$ decreases as $z(\mathbf{V},\mathbf{W}^*)$ decreases. Moreover, the left sides of constraints (19) and (24) do not depend on special values of $\mathbf{W}^*$ and are determined only by the set $\mathcal{S}_{L\backslash K}^c \cup \mathcal{S}_K$. This implies that we do not need to know an optimal value of the vector $\mathbf{W}^* = \mathbf{W}_o^*(K)$. It is enough to know that this value belongs to the set $\mathcal{S}_{L\backslash K}^c \cup \mathcal{S}_K$ (this allows us to construct the K-th constraint in (24)) and makes $z(\mathbf{V},\mathbf{W}^*)$ and $\underline{R}(\mathbf{W}^*)$ to be minimal for at least one $\mathbf{W}^* \in \mathcal{S}_{L\backslash K}^c \cup \mathcal{S}_K$. Therefore, constraints (22) have to be replaced by constraints

$$\mathbb{E}_{p_Y} h_i(Y) \in \begin{cases} S_i, & i \in K \\ S_i^c, & i \in L\backslash K \end{cases}, \ i \in L, \tag{25}$$

where intervals S_i, S_i^c are defined by the set $\mathcal{S}_{L\backslash K}^c \cup \mathcal{S}_K$.

Indeed, $\inf_{\mathbf{W}^* \in \mathcal{S}_{L\backslash K}^c \cup \mathcal{S}_K} \underline{R}(\mathbf{W}^*)$ corresponds to $\inf_{\mathbf{W}^* \in \mathcal{S}_{L\backslash K}^c \cup \mathcal{S}_K} \inf_{\mathcal{P}_J} z(\mathbf{V},\mathbf{W}^*)$. At the same time, this is equivalent to the problem $\inf_{\mathcal{P}_J} z(\mathbf{V},K)$ subject to

$$\mathbb{E}_{p_X} f_i \in \begin{cases} T_i, & i \in J \\ T_i^c, & i \in N\backslash J \end{cases}, \ i \in N, \ \mathbb{E}_{p_Y} h_i = \begin{cases} S_i, & i \in K \\ S_i^c, & i \in L\backslash K \end{cases}, \ i \in L,$$

because constraints (25) contain all points $\mathbf{W}^* \in \mathcal{S}_{L\backslash K}^c \cup \mathcal{S}_K$ and $\inf_{\mathcal{P}_J} z(\mathbf{V},\mathbf{W}^*)$ is achieved at p_Y satisfying one of the values $\mathbf{W}^*$.

So, $z(\mathbf{V},\mathbf{W}^*)$ and $\underline{R}(\mathbf{W}^*)$ can be replaced by $z(J,K)$ and $\underline{R}(K)$. This means that values of $\mathbf{V}$ and $\mathbf{W}$ are taken from the sets $\mathcal{T}_{N\backslash J}^c \cup \mathcal{T}_J$ and $\mathcal{S}_{L\backslash K}^c \cup \mathcal{S}_K$, respectively.

It is worth noticing that this subtle technique allows us to solve a problem of consistency of judgements (22). It is obvious that constraints (22) may be inconsistent by some values of w_i^*, and it is not clear what to do in this case. After introducing constraints (25), the inconsistency means that the corresponding constraint in (24) is removed from the list of constraints to problem (23)-(24).

6 Algorithm for Computing $\underline{R}$

Let us write a final algorithm for computing $\underline{R}$.

Step 1. Choosing a set $\mathcal{S}^c_{L\setminus K_i} \cup \mathcal{S}_{K_i}$ from the possible sets $\mathcal{S}^c_{L\setminus K} \cup \mathcal{S}_K$, $K \subseteq L$.

Step 2. Choosing a set $\mathcal{T}^c_{N\setminus J_j} \cup \mathcal{T}_{J_j}$ from the possible sets $\mathcal{T}^c_{N\setminus J} \cup \mathcal{T}_J$, $J \subseteq N$.

Step 3. Solving the optimization problem with objective function (20) and constraints (21) and (25) by T_i and S_i taken from sets $\mathcal{T}^c_{N\setminus J_i} \cup \mathcal{T}_{J_i}$ and $\mathcal{S}^c_{L\setminus K_i} \cup \mathcal{S}_{K_i}$ defined on Steps 1 and 2, respectively. The result of this step is the value $z(J_j, K_i)$. If $z(J, K_i)$ are obtained for all possible $J \subseteq N$, then go to Step 4, else go to Step 2.

Step 4. Solving linear programming problem (11)-(12) by using the consistent values of $z(J, K_i)$ computed on Step 3. The result of this step is the value $\underline{R}(K)$. If $\underline{R}(K)$ are obtained for all possible $K \subseteq L$, then go to Step 5, else go to Step 1.

Step 5. Solving linear programming problem (23)-(24) by using the consistent values of $\underline{R}(K)$ computed on Step 4. The result of this step is $\underline{R}$.

According to the algorithm, it is necessary to solve $2^l + 1$ linear programming problems (Steps 4 and 5) and 2^{nl} non-linear optimization problems (Step 3). Step 3 can be realized by means of results given in [19]. For solving this non-linear problem in a case of arbitrary judgements, a software program has been developed.

7 Numerical Example 1

Suppose that two experts provide probabilities of events concerning the stress and strength. The first expert: 0.9 and 1 are bounds for the probability that the stress is less than $x_1 = 18$. The second expert: 0 and 0.2 are bounds for the probability that the strength is less than $y_1 = 14$; 0.75 and 1 are bounds for the probability that the strength is less than $y_2 = 20$. The beliefs to experts are 0.9 and $[0.6, 0.8]$, respectively. The beliefs $[a, b]$ mean that the expert provides between $a\%$ and $b\%$ of true judgements. This information can be formally represented as

$$\Pr\{0.9 \le \mathbb{E}I_{[0,18]}(X) \le 1\} = 0.9,$$
$$\Pr\{0 \le \mathbb{E}I_{[0,14]}(Y) \le 0.2\} \in [0.6, 0.8],$$
$$\Pr\{0.75 \le \mathbb{E}I_{[0,20]}(Y) \le 1\} \in [0.6, 0.8].$$

Here $N = \{1\}$, $L = \{1, 2\}$. Let us find $\underline{R} = \underline{\mathbb{E}}\mathbb{E}I_{[0,\infty)}(Y - X)$. Define sets

$$K = \{1,2\},\ \mathcal{S}^c_{L\setminus K} \cup \mathcal{S}_K = \{S_1, S_2\} = \{[0,0.2],[0.75,1]\},$$
$$K = \{1\},\ \mathcal{S}^c_{L\setminus K} \cup \mathcal{S}_K = \{S_1, S_2^c\} = \{[0,0.2],[0,0.75]\},$$
$$K = \{2\},\ \mathcal{S}^c_{L\setminus K} \cup \mathcal{S}_K = \{S_1^c, S_2\} = \{[0.2,1],[0.75,1]\},$$
$$K = \{\varnothing\},\ \mathcal{S}^c_{L\setminus K} \cup \mathcal{S}_K = \{S_1^c, S_2^c\} = \{[0.2,1],[0,0.75]\}.$$

and

$$J=\{1\},\ \mathcal{T}^c_{N\backslash J}\cup\mathcal{T}_J=\{T_1\}=\{[0.9,1]\},$$
$$J=\{\varnothing\},\ \mathcal{T}^c_{N\backslash J}\cup\mathcal{T}_J=\{T_1^c\}=\{[0,0.9]\}.$$

Let us compute $z(J,K)$ for each K and J. According to [19], there holds

$$z=\underline{t}_1(1-\overline{s}_{j(1)}),\ j(i)=\min\{j:x_i\leq y_j\}.$$

Hence $j(1)=2$, and the following hold for $J=\{1\}\subseteq\{1\}$

$$K=\{1,2\},\ z(J,K)=0,$$
$$K=\{1\},\ z(J,K)=0.225,$$
$$K=\{2\},\ z(J,K)=0,$$
$$K=\{\varnothing\},\ z(J,K)=0.225.$$

If $J=\{\varnothing\}$, then $z(J,K)=0$ for all $K\subseteq\{1,2\}$ because $\inf T_1^c=0$. Let us solve problem (11)-(12) for each $K\subseteq L$. For example, if $K=\{1\}$, then

$$\underline{R}(\{1\})=\sup\left(c_0+0.9c_1-0.9d_1\right),$$

subject to $c_1,d_1\in\mathbb{R}_+$, $c_0\in\mathbb{R}$, $c_0+(c_1-d_1)\leq 0.225$, $c_0\leq 0$.

Hence $\underline{R}(\{1\})=0.2025$. Similarly, we can get $\underline{R}(\{1,2\})=0$, $\underline{R}(\{2\})=0$, $\underline{R}(\{\varnothing\})=0.2025$. Let us solve problem (23)-(24)

$$\underline{R}=\sup\left(c_0+0.6c_1-0.8d_1+0.6c_2-0.8d_2\right),$$

subject to $c_1,d_1,c_2,d_2\in\mathbb{R}_+$, $c_0\in\mathbb{R}$,

$$c_0+(c_1-d_1)+(c_2-d_2)\leq 0,$$
$$c_0+(c_1-d_1)\leq 0.2025,$$
$$c_0+(c_2-d_2)\leq 0,\ c_0\leq 0.2025.$$

Hence $\underline{R}=0.0405$. The upper stress-strength reliability $\overline{R}=0.9996$ can be computed in the same way by taking into account that there holds $z=1-\underline{s}_1(1-\overline{t}_1)$.

How to use the obtained interval? This depends on a decision maker and the system purposes (consequences of failures). The values 0.0405 and 0.9996 can be interpreted as pessimistic and optimistic assessments of the stress-strength reliability, respectively. If consequences of the system failure are catastrophic (transport systems, nuclear power plants), then the lower bound (pessimistic decision) for the system reliability has to be determinative and is compared with a required level of the system reliability. If the system failure does not imply major consequences, then the upper bound (optimistic decision) can be used. Generally, the decision maker may use a caution parameter η [1] on the basis of his (her) own experience, various conditions of the system functioning, etc. In this case, the precise value of the system reliability is determined as the linear combination $\eta\underline{R}+(1-\eta)\overline{R}$. At the same time, it can be seen from the example that the obtained interval $[\underline{R},\overline{R}]$ is very wide and the results are too imprecise to make a useful decision concerning the reliability.

8 Numerical Example 2

Suppose that information about the stress and strength is represented as the following set of confidence intervals for two moments: the first and second moments of the stress are in intervals $[7,8]$ and $[40,50]$, respectively, with the confidence probability 0.95; the first and second moments of the strength are in intervals $[12,13]$ and $[150,160]$, respectively, with the confidence probability 0.9. By assuming that all values of the stress and strength are in the interval $[0,50]$ (the sample space), this information can be formally represented as

$$\Pr\{7 \le \mathbb{E}X \le 8\} \in [0.95,1], \ \Pr\{40 \le \mathbb{E}X^2 \le 50\} \in [0.95,1],$$
$$\Pr\{12 \le \mathbb{E}Y \le 13\} \in [0.9,1], \ \Pr\{150 \le \mathbb{E}Y^2 \le 160\} \in [0.9,1].$$

Here $N=\{1,2\}$, $L=\{1,2\}$. Results of computing $z(J,K)$ for each K and J are shown in Table 1.

Table 1: Values of $z(J,K)$

	$K=\{1,2\}$	$K=\{1\}$	$K=\{2\}$	$K=\{\varnothing\}$
$J=\{1,2\}$	0.62	0.122	0.04	0
$J=\{1\}$	0.265	0.085	0.03	0
$J=\{2\}$	0.5	0.12	0.042	0
$J=\{\varnothing\}$	0	0	0	0

Let us solve (11)-(12) for each $K\subseteq L$. For example, if $K=\{1,2\}$, then

$$\underline{R}(\{1,2\}) = \sup\left(c_0+0.95c_1-1d_1+0.95c_1-1d_1\right),$$

subject to $c_1,d_1,c_2,d_2\in\mathbb{R}_+$, $c_0\in\mathbb{R}$,

$$c_0+(c_1-d_1)+(c_2-d_2)\le 0.62,$$
$$c_0+(c_1-d_1)\le 0.265,$$
$$c_0+(c_2-d_2)\le 0.5, \ \ c_0\le 0.$$

Hence $\underline{R}(\{1,2\})=0.589$. Similarly, we can get $\underline{R}(\{1\})=0.116$, $\underline{R}(\{2\})=0.038$, $\underline{R}(\{\varnothing\})=0$. Let us solve problem (23)-(24)

$$\underline{R} = \sup\left(c_0+0.9c_1-1d_1+0.9c_2-1d_2\right),$$

subject to $c_1, d_1, c_2, d_2 \in \mathbb{R}_+$, $c_0 \in \mathbb{R}$,

$$\begin{aligned} c_0 + (c_1 - d_1) + (c_2 - d_2) &\leq 0.589, \\ c_0 + (c_1 - d_1) &\leq 0.116, \\ c_0 + (c_2 - d_2) &\leq 0.038, \quad c_0 \leq 0. \end{aligned}$$

Hence $\underline{R} = 0.487$. The upper bound is $\overline{R} = 1$. If we assume that the intervals for moments of the stress and strength have probabilities 1 (the first-order model), then lower and upper bounds for the stress-strength reliability are 0.62 and 1, respectively.

9 Conclusion

The efficient algorithm for computing the stress-strength reliability by the second-order initial information about the stress and strength has been proposed in the paper. This algorithm uses the imprecise stress-strength reliability models obtained in [19]. Its main virtue is that complex non-linear optimization problem (5)-(6) is reduced to a finite set of simple problems whose solution presents no difficulty. Therefore, this approach might be a basis for developing similar algorithms for reliability analysis of various systems where random variables describing the system reliability behavior are independent. The upper bound for the stress-strength reliability can be similarly computed. In this case, the "inf" is replaced by "sup" in optimization problems and vice versa.

It should be noted also a shortcoming of the algorithm. The joint judgements about the stress and strength can not be used because optimization problem (5)-(6) in this case can not be decomposed into a set of linear programming problems. Therefore, further study is needed to develop methods and efficient algorithms for processing the second-order imprecise probabilities by this type of initial information.

Acknowledgement

This work was supported by the Alexander von Humboldt Foundation. I am very grateful to Prof. Dr. Kurt Weichselberger (Munich University, Germany) for his scientific support and help. I have would like to thank Dr. Anton Wallner (Munich University, Germany) for a very useful discussion and comments related to the independence conditions. I also thank three referees for useful and detailed suggestions that improved the paper.

References

[1] T. Augustin. On decision making under ambiguous prior and sampling information. In G. de Cooman, T.L. Fine, and T. Seidenfeld, editors, *Imprecise Probabilities and Their Applications. Proc. of the 2nd Int. Symposium ISIPTA'01*, pages 9–16, Ithaca, USA, June 2001. Shaker Publishing.

[2] J.O. Berger. *Statistical Decision Theory and Bayesian Analysis*. Springer-Verlag, New York, 1985.

[3] I. Couso, S. Moral, and P. Walley. Examples of independence for imprecise probabilities. In G. de Cooman, F.G. Cozman, S. Moral, and P. Walley, editors, *ISIPTA '99 - Proceedings of the First International Symposium on Imprecise Probabilities and Their Applications*, pages 121–130, Zwijnaarde, Belgium, 1999.

[4] G. de Cooman. Precision–imprecision equivalence in a broad class of imprecise hierarchical uncertainty models. *Journal of Statistical Planning and Inference*, 105(1):175–198, 2002.

[5] G. de Cooman and P. Walley. A possibilistic hierarchical model for behaviour under uncertainty. *Theory and Decision*, 52(4):327–374, 2002.

[6] L. Ekenberg and J. Thorbiörnson. Second-order decision analysis. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 9:13–38, 2 2001.

[7] P. Gärdenfors and N.-E. Sahlin. Unreliable probabilities, risk taking, and decision making. *Synthese*, 53:361–386, 1982.

[8] I.J. Good. Some history of the hierarchical Bayesian methodology. In J.M. Bernardo, M.H. DeGroot, D.V. Lindley, and A.F.M. Smith, editors, *Bayesian Statistics*, pages 489–519. Valencia University Press, Valencia, 1980.

[9] I.O. Kozine and L.V. Utkin. Constructing coherent interval statistical models from unreliable judgements. In E. Zio, M. Demichela, and N. Piccini, editors, *Proceedings of the European Conference on Safety and Reliability ESREL2001*, volume 1, pages 173–180, Torino, Italy, September 2001.

[10] I.O. Kozine and L.V. Utkin. Processing unreliable judgements with an imprecise hierarchical model. *Risk Decision and Policy*, 7(3):325–339, 2002.

[11] V. P. Kuznetsov. *Interval Statistical Models*. Radio and Communication, Moscow, 1991. in Russian.

[12] R. F. Nau. Indeterminate probabilities on finite sets. *The Annals of Statistics*, 20:1737–1767, 1992.

[13] H.T. Nguyen, V. Kreinovich, and L. Longpre. Second-order uncertainty as a bridge between probabilistic and fuzzy approaches. In *Proceedings of the 2nd Conference of the European Society for Fuzzy Logic and Technology EUSFLAT'01*, pages 410–413, England, September 2001.

[14] C.P. Robert. *The Bayesian Choice*. Springer, New York, 1994.

[15] M.C.M. Troffaes and G. de Cooman. Lower previsions for unbounded random variables. In P. Grzegorzewski, O. Hryniewicz, and M.A. Gil, editors, *Soft Methods in Probability, Statistics and Data Analysis*, pages 146–155. Phisica-Verlag, Heidelberg, New York, 2002.

[16] L.V. Utkin. A hierarchical uncertainty model under essentially incomplete information. In P. Grzegorzewski, O. Hryniewicz, and M.A. Gil, editors, *Soft Methods in Probability, Statistics and Data Analysis*, pages 156–163. Phisica-Verlag, Heidelberg, New York, 2002.

[17] L.V. Utkin. Imprecise second-order hierarchical uncertainty model. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 11(3), 2003. To Appear.

[18] L.V. Utkin. A second-order uncertainty model for the calculation of the interval system reliability. *Reliability Engineering and System Safety*, 79(3):341–351, 2003.

[19] L.V. Utkin and I.O. Kozine. Stress-strength reliability models under incomplete information. *International Journal of General Systems*, 31(6):549 – 568, 2002.

[20] L.V. Utkin and I.O. Kozine. Structural reliability modelling under partial source information. In H. Langseth and B. Lindqvist, editors, *Proceedings of the Third Int. Conf. on Mathematical Methods in Reliability (Methodology and Practice)*, pages 647–650, Trondheim, Norway, June 2002. NTNU.

[21] P. Walley. *Statistical Reasoning with Imprecise Probabilities*. Chapman and Hall, London, 1991.

[22] P. Walley. Statistical inferences based on a second-order possibility distribution. *International Journal of General Systems*, 9:337–383, 1997.

[23] K. Weichselberger. *Elementare Grundbegriffe einer allgemeineren Wahrscheinlichkeitsrechnung*, volume I Intervallwahrscheinlichkeit als umfassendes Konzept. Physika, Heidelberg, 2001.

Lev Utkin is with the Department of Computer Science, St.Petersburg Forest Technical Academy, Institutski per. 5, 194021 St.Petersburg, Russia.
Phone +7/812/2476937
E-mail: lvu@utkin.usr.etu.spb.ru